BPS Explained II: Calculating the Equilibration Rate in the Extreme Quantum Limit

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Abstract

This is the second in a series of two lectures on the technique of dimensional continuation, a new method for analytically calculating certain energy transport quantities in a weakly to moderately coupled plasma. Recently, this method was employed by Brown, Preston, and Singleton (BPS) to calculate the electron-ion temperature equilibration rate and the charged particle stopping power to leading and next-to-leading order in the plasma coupling. The basic idea is very simple. Concentrating upon the equilibration rate, the calculation consists of the following two steps: (i) perturbatively expand the rate in the form $d\mathcal{E}/dt = -A\,g^2\ln g + Bg^2 + \mathcal{O}(g^3)$, with the dimensionless expansion parameter being defined by $g = e^2\kappa_e/4\pi T_e$; (ii) analytically calculate the coefficients A and B using the method of dimensional continuation. The factor of 4π should be omitted from g in nonrationalized electrostatic units. In the first lecture, I presented a basic overview of the requisite theoretical machinery of dimensional continuation imported from particle physics, but in a self-contained manner that assumed no familiarity with quantum field theory. In this lecture, I develop the framework further, and then explicitly calculate the electron-ion temperature equilibration rate in the high temperature limit. In this extreme quantum limit, the calculation of the coefficients A and B simplifies considerably, allowing us to concentrate on the physics of the method rather than the added complexity of the more general BPS calculation. This method captures *all* short and long distance physics to second order in g, while three-body and higher correlations are contained in the cubic and higher order terms denoted by $\mathcal{O}(g^3)$. In a weakly to moderately coupled plasma, where g is small, the error term $\mathcal{O}(g^3)$ in this calculation is also small compared to the A- and B-terms, in which case the BPS methodology is quite accurate. Should higher order contributions be required, they can be calculated systematically, thereby improving the accuracy of the result in a controlled manner. To get a feel for the numbers, one finds $g \sim 0.04$ at the center of the sun, where the plasma conditions are $n \sim 5 \times 10^{25}\,\mathrm{cm}^{-3}$ and $T \sim 1\,\mathrm{keV}$. The coupling constant g can be scaled to other plasma regimes through the proportionality relation $g \propto n^{1/2}\,T^{-3/2}$. Of course the application of interest determines the relevant plasma regime, which may or may not lie within the domain of applicability of the BPS calculation. For example, the technique breaks down for warm dense matter where is g not very small; however, this analytic perturbative technique is applicable for ignition in inertial confinement fusion and for other processes in hot a weakly coupled plasma.

Contents

I.   INTRODUCTION AND REVIEW

This is the second lecture on dimensional continuation, a new technique [1] recently used to calculate the charged particle stopping power and the temperature equilibration rate in a weakly to moderately coupled plasma [2]. In Lecture I [3] of this series, I discussed the basic theoretical machinery of dimensional continuation, and I performed a model calculation of the equilibration rate. Reference [4] also contains a summary of the method in a very readable form. In this lecture, I will present the complete calculation of the electron-ion temperature equilibration rate in the *extreme* quantum limit, valid to leading and next-to-leading order in the number density (a more general calculation is performed in Section 12 of Ref. [2] to *all* orders in quantum mechanics, thereby providing an exact interpolation between the extreme classical and quantum limits). This calculation is near exact for a weakly coupled plasma, and it is quite accurate for a moderately coupled plasma. Before proceeding directly to the calculation, however, it might be useful to quickly review some of the more salient features of dimensional continuation discussed in Lecture I.

Under most circumstances, a plasma is not produced in thermal equilibrium; for example, when a laser ionizes a substance, it preferentially heats the electrons over the ions. However, since the electrons are so light, they rapidly come into thermal equilibrium among themselves with temperature T_e; some time later, the ions too will equilibrate among themselves to a common temperature T_I. Finally, the electrons and ions will begin to equilibrate, and it is this process upon which we shall focus. Let $d\mathcal{E}_{eI}/dt$ denote the rate per unit volume at which the electron system at temperature T_e exchanges energy with the ion system at temperature T_I through Coulomb interactions (throughout these notes, I will always measure temperature in energy units). The electron-ion equilibration rate is proportional to the temperature difference between the electrons and ions, and can be expressed by

$$\frac{d\mathcal{E}_{eI}}{dt} = -\mathcal{C}_{eI}\left(T_e - T_I\right) \ . \tag{1.1}$$

To restate the goal of this lecture more precisely, we shall calculate $\mathcal{C}_{eI}$ in the high temperature limit [where two-body scattering is accurately given by the Born approximation], and we will do so *exactly* to leading and next-to-leading order in the plasma coupling parameter g [defined in Lecture I, or in Eq. (2.1) of this lecture]. Under these conditions, the result takes a particularly simple form [2]:

$$\mathcal{C}_{eI} = \frac{\omega_I^2}{2\pi}\,\kappa_e^2\,\sqrt{\frac{m_e}{2\pi\,T_e}}\,\ln\Lambda_{\mathrm{BPS}} \ , \quad \text{with} \quad \ln\Lambda_{\mathrm{BPS}} = \frac{1}{2}\left[\ln\left\{\frac{8T_e^2}{\hbar^2\omega_e^2}\right\} - \gamma - 1\right] \ , \tag{1.2}$$

where $\gamma = 0.57721 \cdots$ is the Euler constant, κ_e and ω_e are the electron Debye wave number and plasma frequency, and $\omega_{\mathrm{I}}^2 = \sum_i \omega_i^2$ is sum of the squares of the ion plasma frequencies.[1]

In the form displayed by equation (1.2), the rate coefficient $\mathcal{C}_{e\mathrm{I}}$ and the Coulomb logarithm $\ln \Lambda_{\mathrm{BPS}}$ do not explicitly depend upon one's choice of electrostatic units, and one may calculate the Debye wave numbers and the plasma frequencies in any desired system. For dimensional continuation, however, it is more convenient to use *rationalized* electrostatic units, and I shall employ this choice from here out. An arbitrary plasma component will be labeled by an index b, and is characterized by mass m_b, charge $e_b = Z_b e$, number density n_b, and temperature T_b. The index b can span the electron and ion plasma components, that is to say, $b = e, i$ with i being an arbitrary ion species. Working in three dimensions for now, the Coulomb potential between two charges e_a and e_b separated by a distance r is $V = e_a e_b / 4\pi r$, and in rationalized units, the Debye wave number and the plasma frequency of species b take the form[2]

$$\kappa_b^2 = \frac{e_b^2 n_b}{T_b} \tag{1.3}$$

$$\omega_b^2 = \frac{e_b^2 n_b}{m_b} . \tag{1.4}$$

The square of the total Debye wave number is $\kappa_{\mathrm{D}}^2 = \sum_b \kappa_b^2$, and the total Debye wave length is $\lambda_{\mathrm{D}} = \kappa_{\mathrm{D}}^{-1}$.

II. CALCULATING THE RATE IN PERTURBATION THEORY

Reference [2], hereafter referred to as BPS, used a double pronged strategy to calculate the rate coefficient (1.2). First, a well chosen [5] dimensionless parameter g was constructed from the relevant dimensionfull plasma quantities, thereby providing a parameter in which to perform a *controlled* perturbative expansion to leading and next-to-leading order in g. The systematic error of the calculation was estimated by the cubic order term in the expansion, which is quite small for a weakly to moderately coupled plasma. While perturbative calculations are not very common in plasma physics, primarily because of the complexity of the systems of interest and the computational focus within the field, the validity of perturbation theory should nonetheless be clear for a "simple" system such as a weakly coupled and fully ionized plasma. The second part of the BPS argument deployed a powerful technique from quantum field theory allowing one to *analytically* calculate the coefficients in the g-expansion.

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[1] Equation (1.2) corresponds to Eqs. (3.61) and (12.12) of Ref. [2], where I have taken this opportunity to correct a small transcription error: when passing from Eq. (12.43) to Eq. (12.44) in Ref. [2], a factor of $1/2$ was dropped. Restoring this factor of $1/2$ changes the additive constant outside the logarithm from the $-\gamma - 2$ that appears in Eq. (12.12) of Ref. [2] to the constant $-\gamma - 1$ in (1.2).

[2] In nonrationalized units, the right-hand-sides of (1.3) and (1.4) should contain an additional factor of 4π.

A. Perturbative Expansions in Weakly Coupled Plasmas

Let us first concentrate on the perturbative expansion. As demonstrated in Ref. [5], and discussed at length in Lecture I [3], for the case at hand the dimensionless plasma coupling parameter is defined by[3]

$$g = \frac{e^2 \kappa_e}{4\pi T_e} . \tag{2.1}$$

Note that g is the ratio of the Coulomb potential energy of two point-charges, separated by the screening length κ_e^{-1}, to the thermal energy of the plasma. Therefore, g can be used to measure the strength of the plasma. To get a feel for the size of g in a hot but not too dense plasma, one finds $g = 0.042$ for a hydrogen plasma under the solar-like conditions $n_e = 5.0 \times 10^{25}\,\mathrm{cm}^{-3}$ and $T_e = 1.3\,\mathrm{keV}$. One can scale to other density and temperature regimes by noting that $g \propto n_e^{1/2} T_e^{-3/2}$. It was shown in Ref. [5] that plasma quantities always expand in *integer* powers of the coupling g, and therefore g is the appropriate parameter in which to perform a controlled perturbative analysis for weakly coupled plasmas.[4] The g-expansion allows for possible non-analytic terms, such as $\ln g$, and in particular, the electron-ion equilibration rate can be written

$$\frac{d\mathcal{E}_{\mathrm{eI}}}{dt} = -\underbrace{A\,g^2 \ln g}_{\mathrm{LO}} + \underbrace{Bg^2}_{\mathrm{NLO}} + \mathcal{O}(g^3) , \tag{2.2}$$

where I have indicated the leading order (LO) and the next-to-leading order (NLO) terms in the expansion. The minus sign on the leading order term of (2.2) is a matter of convention, and for small values of g it renders the coefficient A positive when the energy exchange is positive. Provided we can calculate the coefficients A and B, then (2.2) will be quite accurate in a weakly to moderately coupled plasma in which g is small. Of course this perturbative approach breaks down for strongly coupled plasmas, those for which the value of g is of order one or greater, since every term in the expansion becomes equally important in such cases. However, unlike a model or an uncontrolled calculation, the BPS calculation informs us of its domain of validity, and it provides an estimate of its own error through the size of g.

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[3] In nonrationalized units we would write $g = e^2\kappa_e/T_e$, with $\kappa_e^2 = 4\pi\,e^2 n_e/T_e$.

[4] The usual plasma parameter Γ is related to the expansion parameter by $g \propto \Gamma^{3/2}$ (with proportionality constant of order unity). Small values of Γ therefore imply small values of g, and one may characterize the *strength* of the plasma by either g or Γ. The proportionality relation above follows from the fact that $g \propto n_e^{1/2}$ and $\Gamma \propto n_e^{1/3}$ (the parameter g is defined in terms of a Debye screening length κ_e^{-1}, while Γ is defined in terms of the inter-particle spacing $n_e^{-1/3}$). Furthermore, since $g \propto n_e^{1/2}$, we may loosely think of the g-expansion as an expansion in the electron number density, as I have done in the first paragraph of this introduction. More precisely, of course, we are expanding in the dimensionless quantity $g \propto e^3\,n_e^{1/2}\,T_e^{-3/2}$. See Ref. [5] for more details, particularly Section 1.1 entitled *Relevant Scales and Dimensionless Parameters*.

B. Calculating the Coefficients of the Expansion

We have now reduced the problem to finding the coefficients A and B of the rate (2.2). The coefficient A was first obtained long ago by Spitzer [9] (and it can be estimated by dimensional analysis alone). The coefficient B, however, was calculated only recently in Ref. [2], which employed a powerful technique from quantum field theory called dimensional regularization, or dimensional continuation as I will call it here. Since this technique is quite subtle and has proven to be somewhat controversial, I should emphasize that the method by which one *chooses* to calculate these coefficients is immaterial, except to the extent that it must contain enough physics to extract the next-to-leading order coefficient B. Techniques other than dimensional continuation could well furnish one with the correct expressions for A and B, and perhaps in a simpler manner. However, the only relevant point here is that *by hook or by crook* we must analytically calculate these coefficients, and dimensional continuation is one method of doing this. [5]

Before turning to the calculation of the coefficients, allow me to make a comment on the relation between the next-to-leading order B-term and the Coulomb logarithm. Writing the leading order coefficient as $K = Ag^2$, and defining the dimensionless coefficient $C = \exp\{-B/A\}$, we can express the rate (2.2) in the form

$$\frac{d\mathcal{E}_{e\mathrm{I}}}{dt} = K \ln \Lambda_{\mathrm{coul}} + \mathcal{O}(g^3) , \quad \text{with} \quad \ln \Lambda_{\mathrm{coul}} = -\ln\{Cg\} . \tag{2.3}$$

Since the Coulomb logarithm means different things to different people,[6] I would like to be quite specific in this lecture. By the words "Coulomb logarithm" I simply mean the term $\ln\Lambda_{\mathrm{coul}}$ defined in (2.3), excluding the cubic and higher order terms. Hence, calculating the next-to-leading order coefficient B is equivalent to determining the dimensionless coefficient C *inside* the Coulomb logarithm. Finding dimensionless constants is usually a difficult problem, particularly since one cannot appeal to dimensional analysis for an estimate. It should therefore not be surprising that the coefficient C varies over an order of magnitude or so across the various models within the literature.

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[5] I have recently been informed [6] that Ref. [7], which is designed to apply to both strongly and weakly coupled plasmas, reproduces the BPS result (1.2) in the limit of weak coupling. As far as I am aware, Refs. [2] and [7] are the only works currently in the literature with a formalism strong enough to extract such next-to-leading order physics from first principles.

[6] Student: What is the Coulomb logarithm? Professor: 10.

III.   CALCULATING IN ARBITRARY DIMENSIONS

Before proceeding directly to the calculation in Sec. IV, let us further develop the basic physics and mathematical machinery necessary to perform calculations in an arbitrary number of dimensions. The motivation for this section is, of course, a thorough exposition of the BPS methodology for calculating Coulomb energy-loss processes in a plasma. However, the material in this section is well known and applicable to a wide variety of other calculations, such as particle decay rates in high energy physics and analytic work in statistical mechanics. For the sake of completeness, however, and to establish some results that will be useful in Sec. IV, I will present a cursory but self-contained treatment here. If this material is familiar, then one may proceed directly to the calculation of the temperature equilibration rate in Sec. IV (given the background material in this section, the calculation itself is less than eight pages in length).

We shall start by developing the hyperspherical coordinate system in ν dimensions, which is a straightforward generalization of three dimensional spherical coordinates. To illustrate the utility of hyperspherical coordinates, I will calculate the hyperarea and hypervolume of several multidimensional objects by exploiting their spherical and cylindrical symmetries. These results will be used quite extensively in the next section. As a physical application, I then develop the multidimensional analog of the scattering cross section, which will allow us to consistently include short-distance quantum scattering effects in the g-expansion (quantum effects manifest themselves through the η-dependence of the coefficients in this expansion). Since we are interested in Coulomb energy exchange, we next examine electrostatics in arbitrary dimensions. From the multidimensional form of Gauss' Law, we shall derive the ν-dimensional Coulomb potential $V_\nu(\mathbf{x})$, and we will see that it depends only upon $r = |\mathbf{x}|$ in such a way as to emphasizes short distance physics when $\nu > 3$ and long distance physics when $\nu < 3$. In $\nu = 3$, the short and long distance physics compete with equal strength, giving an infrared and an ultraviolet divergence, and this is what renders the temperature equilibration problem so difficult. To employ the extreme quantum limit, in which the Born approximation for the two-body scattering dominates, we must calculate the Fourier transform of the Coulomb potential in ν dimensions. Interestingly, we shall find that the Fourier transform of $V_\nu(r)$ is given by the quite simple expression $\tilde{V}_\nu(\mathbf{k}) = 1/k^2$, the form of which does not depend upon the dimension of space, but only upon the length of the wavenumber $k = |\mathbf{k}|$. The fact that $\tilde{V}_\nu(k)$ is so simple greatly facilitates calculations in the extreme quantum limit. With potential in hand, we shall then construct kinetic equations in ν dimensions. These equations are explicitly finite in all but $\nu = 3$ dimensions, and I will explain the manner by which the BBGKY hierarchy reduces to the Boltzmann equation and the Lenard-Balescu equation (in $\nu > 3$ and $\nu < 3$ respectively).

1. *Hyperspherical Coordinates*

Kinematic quantities such as the ν-dimensional momentum or position vectors are elements of the same ν-dimensional Euclidean space $\mathbb{R}^\nu$. For definiteness, I will specialize to the case of position $\mathbf{r}$, with the understanding that this vector could also refer to momentum or wavenumber. We can decompose any vector $\mathbf{r} \in \mathbb{R}^\nu$ in terms of a rectilinear orthonormal basis $\hat{\boldsymbol{e}}_\ell$, so that $\mathbf{r} = \sum_{\ell=1}^\nu x_\ell \, \hat{\boldsymbol{e}}_\ell$, or in component notation $\mathbf{r} = (x_1, \cdots, x_\nu)$. Each component is given by $x_\ell = \hat{\boldsymbol{e}}_\ell \cdot \mathbf{r}$, and a change $d\mathbf{r}$ in the vector $\mathbf{r}$ corresponds to a change $dx_\ell = \hat{\boldsymbol{e}}_\ell \cdot d\mathbf{r}$ in the rectilinear coordinate x_ℓ. Letting $\mathbf{r}$ vary successively along the independent directions $\hat{\boldsymbol{e}}_\ell$, we can trace out a small ν-dimensional hypercube with sides of length dx_ℓ; therefore, the rectilinear volume element is given by the simple form

$$d^\nu x = \prod_{\ell=1}^\nu dx_\ell = dx_1 \, dx_2 \cdots dx_\nu \ . \tag{3.1}$$

In performing integrals over the kinematic variables, however, symmetry usually dictates the use of hyperspherical coordinates rather than rectilinear coordinates. I will therefore review the hyperspherical coordinate system in this subsection, deriving the measure for a ν-dimensional volume element $d^\nu x$ in terms of hyperspherical coordinates. For our purposes, the primary utility of hyperspherical coordinates is that the volume element $d^\nu x$ can be written as a product of certain conveniently chosen dimensionless angles, which I will collectively refer to as $d\Omega_{\nu-1}$, and an overall dimensionfull radial factor $r^{\nu-1} \, dr$, so that $d^\nu x = d\Omega_{\nu-1} \, r^{\nu-1} dr$.

Starting with the usual 3-dimensional spherical coordinates of Fig. 1, let us recall why the three dimensional volume element takes the form $d^3x = \sin\theta \, d\theta \, d\phi \, r^2 dr$ (with $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$, and of course $0 \leq r < \infty$; the coordinate singularities of the spherical system are not important here). As depicted in the figure, the three dimensional vector $\mathbf{r}$ has length r, and subtends a polar angle θ relative to the z-axis, while its projection onto the x-y plane subtends an azimuthal angle ϕ relative to the x-axis. The two angles θ and ϕ specify completely the direction of the unit vector $\hat{\mathbf{r}}$. As we increase the polar angle θ by a small amount $d\theta$, the vector $\mathbf{r}$ sweeps out an arc of length $dR_1 = r d\theta$; similarly, a change $d\phi$ in the azimuthal angle will cause $\mathbf{r}$ to sweep out a perpendicular arc (in the x-y plane) of length $dR_2 = r \sin\theta \, d\phi$. Note that the factor of $\sin\theta$ in dR_2 arises from the projection of $\mathbf{r}$ onto the x-y plane. We can make one more independent displacement by moving dr units in the radial direction, which results in a line of length $dR_3 = dr$. For small displacements in $d\theta$,
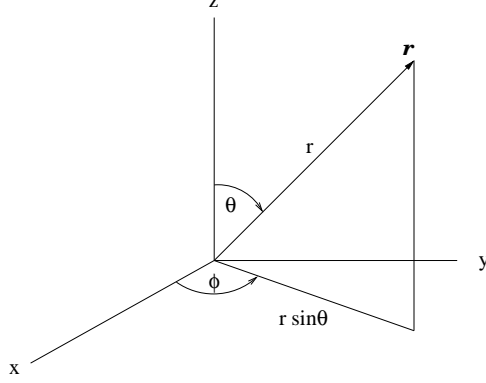
FIG. 1: Spherical coordinates r, θ, ϕ of a point $\mathbf{r}$ in three dimensional space: radial distance r, polar angle θ, and azimuthal angle ϕ. The angles range over the values $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.
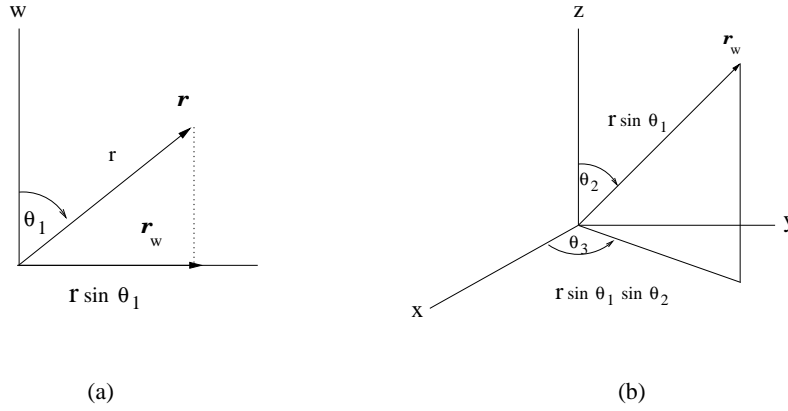

(a)   (b)

FIG. 2: Hyperspherical coordinates $r, \theta_1, \theta_2, \theta_3$ of a point $\mathbf{r}$ in four dimensional space. As before, $r = |\mathbf{r}|$ is the radial distance. The angles are defined as follows. (a) First, let θ_1 be the angle between $\mathbf{r}$ and the w-axis. Let us now project $\mathbf{r}$ onto the orthogonal three dimensional space, so that $\mathbf{r} = (x, y, z, w) \rightarrow \mathbf{r}_w = (x, y, z, 0)$. The length of this projection is $r_w = r \sin \theta_1$, and the projection itself is the same as projecting $\mathbf{r}$ onto the three dimensional hyperplane $w = 0$. (b) The vector $\mathbf{r}_w$ can be viewed as a three dimensional vector $\mathbf{r}_w = (x, y, z)$, which then defines the usual polar and azimuthal angles of Fig. 1, denoted here by θ_2 and θ_3 respectively.

$d\phi$, and dr, the vector $\mathbf{r}$ sweeps out a small cubic volume element with sides of length dR_1, dR_2, and dR_3. The volume of this element is therefore $d^3x = dR_1 \, dR_2 \, dR_3 = rd\theta \cdot r \sin \theta d\phi \cdot dr$.

Let us now consider the volume element d^4x in four dimensional space. Denote the coordinates of a vector $\mathbf{r}$ by x, y, z, w, that is to say, take $\mathbf{r} = (x, y, z, w)$. Since we cannot visualize four dimensional space,[7] let us examine this problem in two steps, each of which

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[7] Apart from visualization problems, we can nonetheless work in higher dimensions by employing analytic geometry and analogies with lower dimensions. For example, an ordinary two-sphere of radius r, which I will denote by $S_2(r)$, has the equation $x^2 + y^2 + z^2 = r^2$ in three dimensional space; a corresponding "three-sphere" $S_3(r)$ in four dimensions can be represented by $x^2 + y^2 + z^2 + w^2 = r^2$. In a similar manner, a two-dimensional plane in three-space can be expressed as $a_1 x + a_2 y + a_3 z = c$ for real numbers a_ℓ and c, while a three-plane in four dimensional space takes the form $a_1 x + a_2 y + a_3 z + a_4 w = c$. As a final example, consider a "three-cone" oriented along the w-axis: $x^2 + y^2 + z^2 - w^2 = 0$. The "conic sections" are obtained by slicing this hypercone with a three-plane along various orientations. For example,

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can be visualized in either two or three dimensions. First, consider the plane that contains the w-axis and the vector $\mathbf{r}$, and let θ_1 be the angle between the w-axis and the vector $\mathbf{r}$ in this plane, as shown in Fig. 2a. We now project $\mathbf{r}$ onto the $w = 0$ hyperplane (a three dimensional slice of four-space), calling the projected vector $\mathbf{r}_w$. Since the three-plane $w = 0$ lies perpendicular to each of the axes x, y, and z, the vector $\mathbf{r}_w$ lies in the three dimensional space shown in Fig. 2b, and its length is $|\mathbf{r}_w| = r \sin\theta_1$. Let the angle θ_2 be the polar angle between the z-axis and the vector $\mathbf{r}_w$, while θ_3 is the usual azimuthal angle ϕ, as illustrated in Fig. 2b. As we vary the three angles and the radial coordinate, we sweep out a four-dimensional cube (or an approximate cube) with sides of length $dR_1 = r\, d\theta_1$, $dR_2 = r\sin\theta_1 d\theta_2$, $dR_3 = r\sin\theta_1 \sin\theta_2\, d\theta_3$, and $dR_4 = dr$. This gives a four dimensional volume element

$$d^4x \equiv dR_1\, dR_2\, dR_3\, dR_4 = \sin^2\theta_1 d\theta_1 \ \sin\theta_2 d\theta_2\ d\theta_3\ r^3\, dr \ , \tag{3.2}$$

where $0 \le \theta_\ell \le \pi$ for $\ell = 1, 2$ and $0 \le \theta_3 < 2\pi$. As a useful exercise, we can find the four dimensional hypervolume enclosed by a three-sphere of radius r by integrating the volume element over the appropriate bounds,

$$B_4 = \int_0^\pi d\theta_1 \sin^2\theta_1 \int_0^\pi d\theta_2 \sin\theta_2 \int_0^{2\pi} d\theta_3 \int_0^r dr'\, r'^3 = \frac{1}{2}\,\pi^2\, r^4 \ . \tag{3.3}$$

The derivative of B_4 with respect to r gives the hypersurface area of the three-sphere,

$$S_3 = \frac{dB_4}{dr} = 2\pi^2\, r^3 \ . \tag{3.4}$$

This is analogous to a three dimensional ball of radius r and volume $B_3 = 4\pi\, r^3/3$ bounded by the two-sphere of area $S_2 = 4\pi r^2$.

We can readily generalize this procedure to an arbitrary number of dimensions. Consider a point $\mathbf{r} \in \mathbb{R}^\nu$ given by the rectilinear coordinates $\mathbf{r} = (x_1, x_2, \cdots, x_\nu)$. Let θ_1 be the angle between the vector $\mathbf{r}$ and the x_1-axis, in a manner similar to that of Figs. 1 and 2a. Note that $dR_1 = rd\theta_1$ is the arc length swept out by $\mathbf{r}$ as the angle θ_1 is incremented by $d\theta_1$. Let us now project $\mathbf{r}$ onto the hyperplane $x_1 = 0$, the $(\nu - 1)$-plane normal to the x_1-axis and passing through the origin, calling this projection $\mathbf{r}_1$: that is to say, let $\mathbf{r} \to \mathbf{r}_1 = (0, x_2, \cdots, x_\nu)$. The length of this vector is $r_1 = r \sin\theta_1$. Let us proceed to the next step and define the angle θ_2 as the angle between the x_2-axis and $\mathbf{r}_1$, in which case, as the angle θ_2 is varied by $d\theta_2$, the vector $\mathbf{r}_1$ sweeps out an arc of length $dR_2 = r_1 d\theta_2 = r\sin\theta_1 d\theta_2$. In a similar

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if we slice the three-cone by the hyperplane $w = r$ orthogonal to the w-axis, then we find a two-sphere $x^2 + y^2 + z^2 = r^2$; if we slice the three-cone by a hyperplane along the z-axis, say $z = r$, then we find the hyperboloid of two sheets $x^2 + y^2 = w^2 - r^2$ oriented along the w-axis.

TABLE I: Solid angle $\Omega_{\nu-1}$ as a function of dimension ν.

ν	2	3	4	5	6	7	8	$\cdots$	20
$\Omega_{\nu-1}$	2π	4π	$2\pi^2$	$8\pi^2/3$	π^3	$16\pi^3/15$	$\pi^4/3$	$\cdots$	$\pi^{10}/181440$
value	6.28	12.6	19.7	26.3	31.0	33.1	32.5	$\cdots$	0.516

fashion, project $\mathbf{r}_1$ onto the x_2-plane, that is, the plane described by $x_1 = 0$ and $x_2 = 0$. This projection is given by $\mathbf{r} \to \mathbf{r}_2 = (0, 0, x_3, \cdots, x_\nu)$, and the length of the projection is $r_2 = r_1 \sin\theta_2 = r \sin\theta_1 \sin\theta_2.$[8] For the general ℓ^{th} iteration, let θ_ℓ be the angle between the x_ℓ-axis and $\mathbf{r}_{\ell-1}$, so that $dR_\ell = r_{\ell-1} d\theta_\ell = r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{\ell-1} d\theta_\ell$. In summary, we define the quantities

$$\theta_\ell = \text{angle between the } x_\ell\text{-axis and } \mathbf{r}_{\ell-1} \tag{3.5}$$

$$dR_\ell = r_{\ell-1} d\theta_\ell = r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{\ell-1} d\theta_\ell \tag{3.6}$$

$$\mathbf{r} \to \mathbf{r}_\ell = (0, \cdots, 0, x_{\ell+1}, \cdots, x_\nu) \tag{3.7}$$

$$r_\ell = r_{\ell-1} \sin\theta_\ell = r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{\ell-1} \sin\theta_\ell \;, \tag{3.8}$$

where we have used the fact that $r_{\ell-1} = r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{\ell-1}$. The last two are lines provide the projection for the $(\ell + 1)^{\text{st}}$ step. This gives the ν-dimensional volume element

$$d^\nu x = \prod_{\ell=1}^{\nu} dR_\ell = \sin^{\nu-2}\theta_1 d\theta_1 \cdot \sin^{\nu-3}\theta_2 d\theta_2 \cdots \sin\theta_{\nu-2} d\theta_{\nu-2} \cdot d\theta_{\nu-1} \cdot r^{\nu-1} dr \;. \tag{3.9}$$

For notational convenience, I will write the angular measure in (3.9) as $d\Omega_{\nu-1}$, so that

$$d^\nu x = d\Omega_{\nu-1} r^{\nu-1} dr \;. \tag{3.10}$$

As we proved in the Lecture I, the integration over all angles gives the total solid angle

$$\Omega_{\nu-1} \equiv \int d\Omega_{\nu-1} = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)} \;, \tag{3.11}$$

and Table I illustrates the numerical values of this solid angle over a wide range of dimensions. Note that $\Omega_{\nu-1}$ reaches a maximum around $\nu = 7$ and then slowly decreases.

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[8] Just for good measure, let us do one more iteration. Let θ_3 be the angle between the x_3-axis and the previous projection $\mathbf{r}_2$, in which case $dR_3 = r_2 d\theta_3 = r \sin\theta_1 \sin\theta_2 d\theta_3$. Let us now project $\mathbf{r}_2$ onto the x_3-plane described by $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$, *i.e.* $\mathbf{r} \to \mathbf{r}_3 = (0, 0, 0, x_4, \cdots, x_\nu)$. The length of this vector is $r_3 = r_2 \sin\theta_3 = r \sin\theta_1 \sin\theta_2 \sin\theta_3$, and the next iteration can begin.

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As a matter of completeness, let us prove (3.11) here. First, consider the one-dimensional Gaussian integral

$$\int_{-\infty}^{\infty} dx \, e^{-x^2} = \sqrt{\pi} \; . \tag{3.12}$$

If we multiply both sides together ν times (with $\nu \in \mathbb{Z}^+$), we find

$$(\sqrt{\pi})^{\nu} = \int_{-\infty}^{\infty} dx_1 \, e^{-x_1^2} \int_{-\infty}^{\infty} dx_2 \, e^{-x_2^2} \cdots \int_{-\infty}^{\infty} dx_{\nu} \, e^{-x_{\nu}^2} = \int d^{\nu}x \, e^{-\mathbf{r}^2} \; , \tag{3.13}$$

where the vector $\mathbf{r}$ in the exponential of the last expression is the ν-dimensional vector $\mathbf{r} = (x_1, x_2, \cdots, x_{\nu})$, and $\mathbf{r}^2 = \mathbf{r} \cdot \mathbf{r} = \sum_{\ell=1}^{\nu} x_{\ell}^2$. As in (3.10), we can factor the angular integrals out of the right-hand-side of (3.13), and the remaining one-dimensional integral can be converted to a Gamma function with the change of variables $t = r^2$:

$$\pi^{\nu/2} = \int d\Omega_{\nu-1} \cdot \int_0^{\infty} dr \, r^{\nu-1} \, e^{-r^2} = \int d\Omega_{\nu-1} \cdot \frac{1}{2} \, \Gamma(\nu/2) \; . \tag{3.14}$$

Solving for $\int d\Omega_{\nu-1}$ in (3.14) gives (3.11).

In calculating the temperature equilibration rate and the charged particle stopping power, we encounter integrals of the form

$$I_1(\nu) \equiv \int d^{\nu}x \, f_1(r) = \Omega_{\nu-1} \int_0^{\infty} dr \, r^{\nu-1} \, f_1(r) \tag{3.15}$$

$$I_2(\nu) \equiv \int d^{\nu}x \, f_2(r, \theta) = \Omega_{\nu-2} \int_0^{\infty} dr \, r^{\nu-1} \int_0^{\pi} d\theta \, \sin^{\nu-2}\theta \, f_2(r, \theta) \; , \tag{3.16}$$

respectively, with $\nu \in \mathbb{Z}^+$. The exact forms of f_1 and f_2 do not concern us here, except that their angular dependence is determined by the following considerations: the integral (3.15) is spherically symmetric since the energy exchange between plasma species is isotropic, while in the latter integral (3.16), the motion of the charged particle defines a preferred direction around which one must integrate. The integrals I_1 and I_2 can be viewed as functions defined on the positive integers, and as discussed at length in Lecture I [3], Carlson's Theorem [8] ensures that there is a unique analytic continuation onto the complex plane. As our first application in this section, let us see how the expressions (3.15) and (3.16) provide a means by which to easily and conveniently perform this analytic continuation to complex values of the spatial dimension ν, thereby rendering ν truly arbitrary. First, the solid angles $\Omega_{\nu-1}$ and $\Omega_{\nu-2}$ are composed of a simple exponential factor $\pi^{\nu/2}$ and a Gamma function, whose analytic continuations have been well studied. As for the integrals, simply treat ν as a complex parameter, performing the one dimensional integral (3.15) and the double integral (3.16) in the usual manner of ordinary calculus. This provides functions $I_1(\nu)$ and $I_2(\nu)$ of a
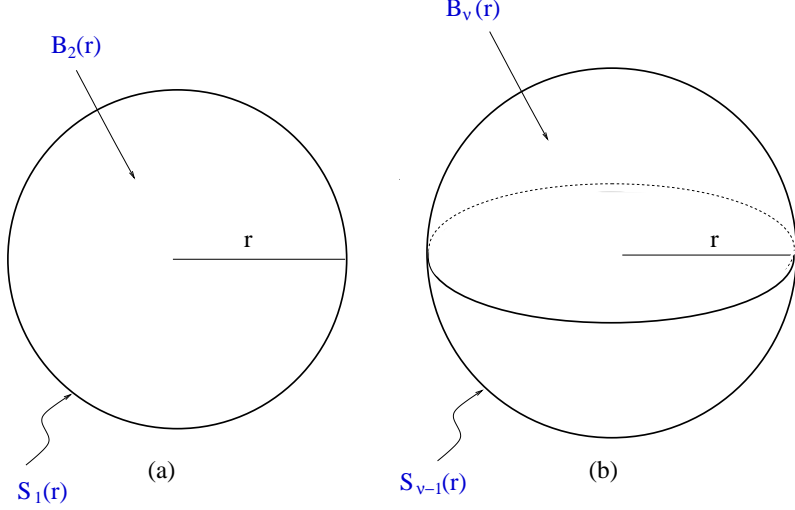
FIG. 3: A $(\nu-1)$-dimensional sphere $S_{\nu-1}$ of radius r bounds the ν-dimensional ball $B_\nu(r)$ of radius r. By integrating over successive shells of area, we can find the volume by $B_\nu(r) = \int_0^r dr' S_{\nu-1}(r')$; or conversely $S_\nu(r) = B_\nu'(r)$.

complex argument $\nu \in \mathbb{C}$, in fulfillment of Carlson's Theorem. Double integrals of the form (3.16) were used extensively in Ref. [2] to calculate the stopping power, where the angle θ is determined by the direction of motion of the charged particle. Calculating the temperature equilibration rate, on the other hand, requires only the simpler one dimensional integral (3.15), as the energy exchange in this process is isotropic.

2. The Hypervolume of Spheres, Disks, and Cylinders

We shall now calculate the hypervolume of several useful geometric objects. Let us first consider a ν-dimensional ball of radius r, defined by the set of points $\mathbf{x} \in \mathbb{R}^\nu$ for which $|\mathbf{x}| \le r$. We will denote this object by $B_\nu(r)$, and in two and three dimensions this is a disk and a spherical, both volume centered at the origin. We can find the ν-dimensional hypervolume of the ball $B_\nu(r)$ by simply integrating (3.9) over all permissible values of the coordinates. It should cause no confusion to denote the hypervolume of the region $B_\nu(r)$ by the same symbol, and using (3.11) we find

$$B_\nu(r) = \int d\Omega_{\nu-1} \int_0^r dr'\, r'^{\nu-1} = \frac{\pi^{\nu/2}}{\Gamma(\nu/2+1)}\, r^\nu \;. \tag{3.17}$$

The boundary of $B_\nu(r)$ is a $(\nu-1)$-dimensional sphere $S_{\nu-1}(r)$ defined by $|\mathbf{x}| = r$, or $\sum_{\ell=1}^\nu x_\ell^2 = r^2$. By differentiating (3.17) with respect to the radius r, we can also find the hyperarea of a $(\nu-1)$-dimensional sphere $S_{\nu-1}(r)$ of radius r in $\mathbb{R}^\nu$,

$$S_{\nu-1}(r) = \frac{dB_\nu}{dr} = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)}\, r^{\nu-1} = \Omega_{\nu-1}\, r^{\nu-1} \;. \tag{3.18}$$
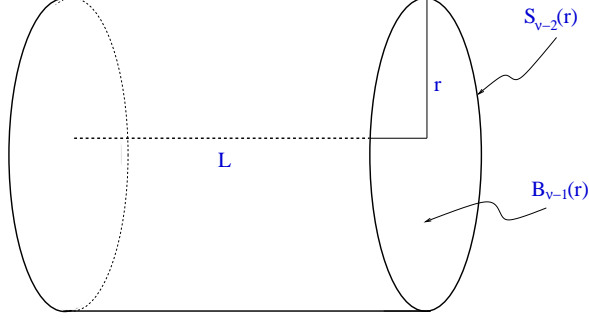
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FIG. 4: The hyperarea of a hypercylinder $C_{\nu-1}(r, L)$ of length L and radius r is $C_{\nu-1}(r, L) = S_{\nu-2}(r) \cdot L$, and the hypervolume bounded by the cylinder is $V_\nu(r, L) = B_{\nu-1}(r) \cdot L$.

For brevity, I have denoted the hyperarea by the same symbol $S_{\nu-1}(r)$ as the sphere itself, which is simply the $(\nu-1)$-dimensional boundary of the region $B_\nu(r)$. This is illustrated in Fig. 3. The distinction I am making between "hypervolume" and "hyperarea" is somewhat arbitrary, since these are both terms involving regions in a higher dimensional space. When I wish to talk about a ν-dimensional subregion of the hyperspace $\mathbb{R}^\nu$, such as $B_\nu(r)$, I will use the term hypervolume. On the other hand, when I wish to emphasize a boundary region of a hypervolume, such as $S_{\nu-1}(r)$, I will use the term "hyperarea." Regarding the usage of the term "solid angle," suppose we keep the radius r fixed but vary the angles θ_i over ranges $d\theta_i$. The region swept out by this procedure lies on the $(\nu-1)$-dimensional sphere $S_{\nu-1}(r)$ with a hyperarea $dS_{\nu-1} = d\Omega_{\nu-1}\, r^{\nu-1}$. We are therefore justified in calling $d\Omega_{\nu-1}$ the solid angle in ν dimensions.

Finally, let us discuss the $(\nu-1)$-dimensional cylindrical $C_{\nu-1}(r, L)$ of radius r and length L. Again, it is easiest to argue from analogy in three dimensions. To form a two-cylinder $C_2(r, L)$ in $\mathbb{R}^3$, we let a two dimensional disk $B_2(r)$ sweep out a volume as it moves a distance L in the orthogonal direction, which is illustrated in Fig. 4a. Similarly, a corresponding $(\nu-1)$-dimensional cylinder is formed by letting a $(\nu-1)$-dimensional ball $B_{\nu-1}(r)$ sweep out a distance L along the orthogonal axis, as illustrated in Fig. 4b. Therefore, the hyperarea of the $(\nu-1)$-dimensional cylinder is

$$C_{\nu-1}(r, L) = S_{\nu-2}(r) \cdot L . \tag{3.19}$$

The ν-dimensional hypervolume enclosed by this cylinder is

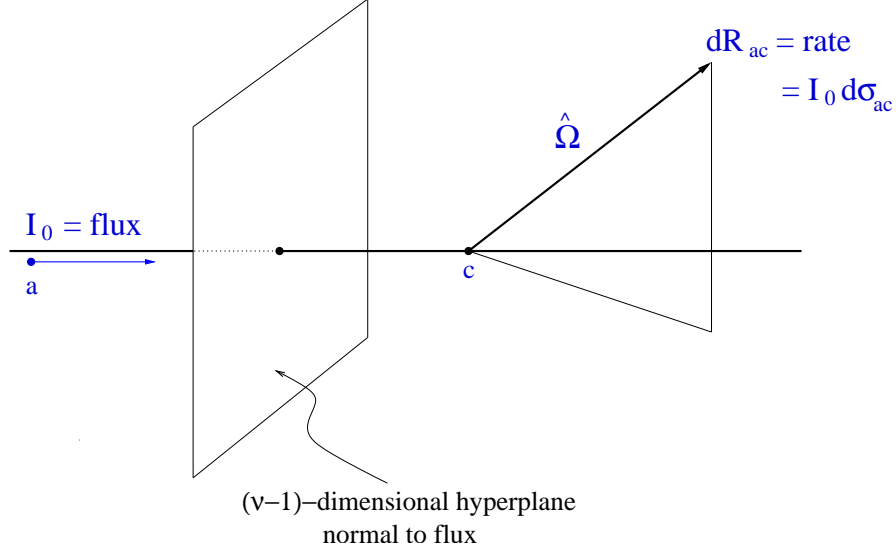$$V_\nu(r, L) = B_{\nu-1}(r) \cdot L . \tag{3.20}$$

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FIG. 5: Definition of the cross section in a general number of dimensions. The incident flux I_0 of species a is the rate of particles per unit hyperarea normal to the flow. The units of I_0 are $L^{1-\nu} \cdot T^{-1}$, where L and T denote the units of space and time. By definition, the differential *cross section* $d\sigma_{ac}$ is related to the rate dR_{ac}, each at angular position $\hat{\Omega}$, by $dR_{ac}(\hat{\Omega}) = I_0 \, d\sigma_{ac}(\hat{\Omega})$. The cross section per unit solid angle about the direction $\hat{\Omega}$ is denoted by $d\sigma_{ac}/d\Omega$. Except for the specification of $\hat{\Omega}$, this definition does not depend upon the dimensionality of space, and the units of $d\sigma_{ac}$ are $L^{\nu-1}$.

B. The Cross Section

As a physical application of hyperspherical coordinates, let us calculate the form of the classical "cross section" in ν-dimensions. For simplicity we will consider a projectile striking a *fixed* target, although we can perform a similar analysis in the center-of-mass frame of the two particles. Such a scattering experiment is illustrated in Fig. 5, in which a beam of incident particles, denoted by the label a, is fired at a target c with incident flux I_0. The rate $dR_{ac}(\hat{\Omega})$ at which the scattered a-particles enter a given solid angle $d\Omega_{\nu-1}$ about the direction $\hat{\Omega}$ is then measured. The flux I_0 is a characterization of the rate at which particles move along the beam axis. In ν dimensions, the spatial region normal to the axis is a $(\nu-1)$-dimensional hyperplane, and the flux I_0 is the number of particles per second per unit hyperarea passing through this plane. For example, if the beam direction is $\hat{\mathbf{n}}$, then the number of particles in a time interval dt passing through a hyperarea $dA_{\hat{\mathbf{n}}}$ normal to $\hat{\mathbf{n}}$ is given by $dN = I_0 \cdot dA_{\hat{\mathbf{n}}} \cdot dt$. The engineering units of I_0 are therefore $L^{1-\nu} \cdot T^{-1}$. In analogy with the usual cross section in three dimensions, we define $d\sigma_{ac}$ through

$$d\sigma_{ac} \cdot I_0 = dR_{ac} \,, \qquad (3.21)$$

and $d\sigma_{ac}$ therefore has engineering units of $L^{\nu-1}$.

Suppose the scattering center is a central force, such as the ν-dimensional Coulomb potential. The particle is confined to a two-dimensional plane for central potential motion, and

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this holds true even in ν dimensions. Let b denote the impact parameter of projectile. As the particle traverses its plane of motion, its position is uniquely characterized by a function $b = b(\theta)$, where θ is the angle between the beam direction and the projectile (with the scattering center defining the origin). From Fig. 4, the number of particles per unit time passing through the hyperannulus of width db and radius b is is $dN = S_{\nu-2}(b)db \cdot I_0 = \Omega_{\nu-2} b^{\nu-2} db \cdot I_0$, and by particle number conservation, the same number of scattered particles reaches the hyperannulus at θ. The cross section in a ν-dimensional central potential is therefore given by

$$d\sigma_{ac} = \Omega_{\nu-2} b^{\nu-2} db .$$ (3.22)

This is Eq. (8.31) of Ref. [2], the starting point for the classical calculation. The cross section will appear in the Boltzmann equation. To include two-body quantum scattering effects, we replace the classical cross section by the quantum cross section:

$$|\mathbf{v}_a - \mathbf{v}_c| d\sigma_{ac} = |T|^2 \frac{d^\nu p_c}{(2\pi\hbar)^\nu} \frac{d^\nu p_a}{(2\pi\hbar)^\nu} ,$$ (3.23)

where T is the quantum scattering amplitude. In the calculations that follow, we shall use work in the extreme quantum limit where the Born approximation for the amplitude can be employed.

C. The Coulomb Potential in Arbitrary Dimensions

Now that we have discussed the cross section in an arbitrary central potential, let us concentrate on the special case of the Coulomb potential. The physics of dimensional continuation is contained in the ν-dependence of the Coulomb potential in ν-dimensional space, which ensures that short distance physics is emphasized in $\nu > 3$ and long distance physics in $\nu < 3$. Changing the spatial dimension about $\nu = 3$ therefore acts as a "physics sieve." Let us first construct the electric field of a point charge in ν dimensions. Maxwell's equations are easily generalized to an arbitrary number of dimensions, and in particular, we can write

$$\boldsymbol{\nabla} \cdot \mathbf{E}(\mathbf{x}) = \rho(\mathbf{x}) ,$$ (3.24)

where $\mathbf{E} = (E_1, \cdots, E_\nu)$ is the electric field vector and $\boldsymbol{\nabla} = (\partial/\partial x_1, \cdots, \partial/\partial x_\nu)$ is the ν-dimensional spatial gradient. The charge density ρ has engineering units of charge divided length to the νth power, which I will write as Q/L^ν. In integral form, the equation can be written

$$\int_\Sigma d^\nu x \, \boldsymbol{\nabla} \cdot \mathbf{E} = e ,$$ (3.25)
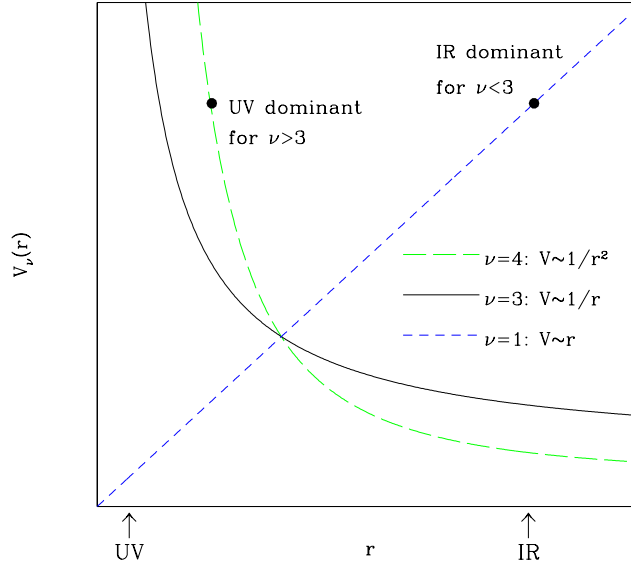
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FIG. 6: Short-distance ultraviolet (UV) physics dominates in dimensions $\nu > 3$. Long-distance or infrared (IR) physics dominates when $\nu < 3$. UV and IR physics are equally important in $\nu = 3$.

where e is the total electric charge contained in the hypervolume Σ. Note that the dimensionality of space is now explicitly indicated by the integration measure. We can employ the usual symmetry argument to find the electric field of a point source at the origin. Let B_r be the ν-dimensional ball of radius r centered on the point charge e, and denote the $(\nu-1)$-dimensional hyperspherical boundary by S_r. By symmetry, the field $\mathbf{E}(\mathbf{x})$ points radially outward with a magnitude $E(r)$ along the direction $\hat{\mathbf{x}}$ normal to S_r. The length $E(r)$ depends only upon the radial distance $r = |\mathbf{x}|$ and not upon its angular location along S_r. The divergence theorem holds in an arbitrary number of dimensions, and since the hyperarea of S_r is given by (3.18), we find:

$$e = \int_{B_r} d^\nu x\, \boldsymbol{\nabla}\cdot\mathbf{E} = \oint_{S_r} d\mathbf{A}\cdot\mathbf{E} = \Omega_{\nu-1}\, r^{\nu-1} \cdot E(r) \quad \text{with} \quad \Omega_{\nu-1} = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)} \; . \tag{3.26}$$

The electric field is therefore given by

$$\mathbf{E}(\mathbf{x}) = \frac{\Gamma(\nu/2)}{2\pi^{\nu/2}} \frac{e}{r^{\nu-1}} \,\hat{\mathbf{x}} \; , \tag{3.27}$$

where we are using the notation $\mathbf{x} = r\,\hat{\mathbf{x}}$, with $\hat{\mathbf{x}}$ being a unit vector pointing in the direction of $\mathbf{x}$.

I find it more convenient to work with the electric potential, a scalar quantity $\phi(r)$ defined by $E(r) = -d\phi(r)/dr$. In fact, I will work with the potential energy $V_\nu = e\,\phi(r)$, so that

$$V_\nu(\mathbf{x}) = \frac{\Gamma(\nu/2 - 1)}{4\pi^{\nu/2}} \frac{e^2}{r^{\nu-2}} \; , \tag{3.28}$$

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where I have appended a subscript to the potential energy to remind us that we are working in ν dimensions. For two charges e_a and e_b separated by a distance r, one only need replace e^2 by the product $e_a e_b$. For $\nu = 3$, the geometric factor in (3.28) becomes $1/4\pi$, which is the origin of the 4π of rationalized units. Figure 6 shows the Coulomb potential for $\nu = 3$, along with two representative dimensions on either side of $\nu = 3$. As the figure illustrates, the short distance behavior of the Coulomb potential becomes more pronounced in higher dimensions, while long-distances are emphasized in lower dimensions. For aesthetic reasons, the arbitrary integration constant for the potential energy has been adjusted in each case so that all three graphs intersect at a single point. This figure illustrates quite dramatically that by simply dialing the dimension ν, we can dial a potential $V_\nu(r)$ that filters either long-distance or short-distance physics.

In the Born approximation to quantum Coulomb scattering, which we shall employ shortly, we need the Fourier transform of the Coulomb potential (3.28). Unlike the spatial representation $V_\nu(\mathbf{x})$, the Fourier representation of the ν-dimensional Coulomb potential takes the same form in any dimension, namely,

$$\tilde{V}_\nu(\mathbf{k}) = -\frac{e^2}{k^2} , \tag{3.29}$$

where $k^2 = \sum_{\ell=1}^{\nu} k_\ell^2$ is just the square of the norm of the ν-dimensional wave number $\mathbf{k}$, and I am using the conventions

$$V_\nu(\mathbf{x}) = \int \frac{d^\nu k}{(2\pi)^\nu} e^{-i\mathbf{x}\cdot\mathbf{k}} \tilde{V}_\nu(\mathbf{k}) \tag{3.30}$$

$$\tilde{V}_\nu(\mathbf{k}) = \int d^\nu x \, e^{i\mathbf{x}\cdot\mathbf{k}} V_\nu(\mathbf{x}) . \tag{3.31}$$

With these conventions, the amplitude in the Born approximation in any dimension is given by

$$T_{\mathrm{B}} = \hbar \frac{e^2}{q^2} , \tag{3.32}$$

where $\mathbf{q} = \mathbf{p}_a - \mathbf{p}_b$ is the momentum transfer during the collision. This is a function only of the square of its argument q^2. In particular, the Born approximation does not introduce dependence upon the center-of-momentum energy W, and this is what renders its use so convenient.

Expression (3.29) for the Fourier transform of the potential (3.28) can be established in a number of ways, perhaps the easiest being an straightforward application of Laplace's equation,

$$\nabla^2 V_\nu(\mathbf{x}) = e^2 \, \delta^{(\nu)}(\mathbf{x}) . \tag{3.33}$$

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Upon inserting (3.30) for $V_\nu(\mathbf{x})$ into (3.33) and using the integral representation of the delta-function, we can write Laplace's equation in the form

$$-\int \frac{d^\nu k}{(2\pi)^\nu}\, e^{-i\mathbf{x}\cdot\mathbf{k}}\, k^2\, \tilde{V}_\nu(\mathbf{k}) = e^2 \int \frac{d^\nu k}{(2\pi)^\nu}\, e^{-i\mathbf{x}\cdot\mathbf{k}}\ , \tag{3.34}$$

and solving for $\tilde{V}(\mathbf{k})$ provides (3.29). It might also be informative to prove (3.29) using the more direct approach of performing the Fourier transform directly. Substituting the Coulomb potential (3.28) into (3.31), and then using (3.16) to rewrite the ν-dimensional integral, we find

$$\tilde{V}_\nu(\mathbf{k}) \;=\; \Omega_{\nu-2} \int_0^\infty dr\, r^{\nu-1} \int_0^\pi d\theta\, \sin^{\nu-2}\theta\, e^{irk\cos\theta} \cdot \frac{\Gamma(\nu/2-1)}{4\pi^{\nu/2}}\, \frac{e^2}{r^{\nu-2}} \tag{3.35}$$

$$=\; \frac{e^2}{2\sqrt{\pi}}\, \frac{\Gamma(\nu/2-1)}{\Gamma(\nu/2-1/2)} \int_0^\infty r\, dr \int_0^1 du\, (1-u^2)^{(\nu-3)/2} \left[e^{ik\,ru} + e^{-ik\,ru} \right]\ , \tag{3.36}$$

where we have made the change of variables $u = \cos\theta$. It is convenient to keep the exponential terms in square brackets rather than converting their sum into a cosine term. We will perform the r-integration by deforming the contour slightly off the real axis,

$$\int_0^\infty dr\, r \left[e^{ik\,ru} + e^{-ik\,ru} \right] \;=\; \int_0^{\infty+i\epsilon} dr\, r\, e^{ik\,ru} + \int_0^{\infty-i\epsilon} dr\, r\, e^{-ik\,ru} \tag{3.37}$$

$$=\; -\frac{2}{(ku)^2}\ . \tag{3.38}$$

Upon substituting this back into (3.36) and changing variables to $t = u^2$ we can write

$$\tilde{V}_\nu(\mathbf{k}) \;=\; \frac{e^2}{2\sqrt{\pi}}\, \frac{1}{k^2}\, \frac{\Gamma(\nu/2-1)}{\Gamma(\nu/2-1/2)} \int_0^1 du\, (1-t)^{(\nu-3)/2}\, t^{-3/2}\ , \tag{3.39}$$

where the second term in the integrand introduces the pole $1/(\nu-3)$ into physical quantities, and the t-integral takes the form of the Euler Beta function

$$B(x,y) = \int_0^1 dt\, t^{x-1}\, (1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{3.40}$$

with $x = -1/2$ and $y = \nu/2 - 1$. Using $\Gamma(-1/2) = -2\sqrt{\pi}$ gives (3.29). In Section IV B we will need yet another representation of the Beta function, which I record here for convenience:

$$B(x,y) = \int_0^\infty dt\, t^{x-1}(1+t)^{-x-y} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\ . \tag{3.41}$$

D.   Kinetic Equations in Arbitrary Dimensions

1.   *Distribution Functions*

A particle in a space $\mathbb{R}^\nu$ of arbitrary dimension $\nu \in \mathbb{Z}^+$ is fully characterized by its position and momentum $\mathbf{x}$ and $\mathbf{p}$, which have rectilinear coordinates x_ℓ and p_ℓ for $\ell = 1, \cdots, \nu$. I will often denote the square and the magnitude of the momentum by $p^2 = \mathbf{p} \cdot \mathbf{p} = \sum_{\ell=1}^\nu p_\ell^2$ and $p = |\mathbf{p}|$, respectively. For example, p^{-3} is shorthand for $|\mathbf{p}|^{-3} = (\sum_{\ell=1}^\nu p_\ell^2)^{-3/2}$. A swarm of particles distributed over position and momentum values is characterized by a distribution function f defined by

$$f(\mathbf{x}, \mathbf{p}, t) \frac{d^\nu x\, d^\nu p}{(2\pi\hbar)^\nu} \equiv \begin{array}{l} \text{number of particles in a hypervolume } d^\nu x \text{ about } \mathbf{x} \\ \text{and } d^\nu p \text{ about } \mathbf{p} \text{ at time } t \, . \end{array} \tag{3.42}$$

The factor of $(2\pi\hbar)^\nu$ in the denominator is a conventional normalization factor, and for a spatially uniform distribution f_a this gives the normalization

$$\int \frac{d^\nu p_a}{(2\pi\hbar)^\nu} f_a(\mathbf{p}_a) = n_a \, , \tag{3.43}$$

where n_a is the number density of a-type particles. That is to say, $n_a\, d^\nu x$ is the number of particles of species a in a hypervolume $d^\nu x$, and the engineering units of n_a are therefore $L^{-\nu}$. From (3.43), we see that a normalized Maxwell-Boltzmann distribution at temperature T_a and number density of n_a is given by

$$f_a(\mathbf{p}_a) = n_a \left(\frac{2\pi\hbar^2 \beta_a}{m_a} \right)^{\nu/2} \exp\left\{ -\beta_a \frac{p_a^2}{2m_a} \right\} = n_a \lambda_a^\nu \, e^{-\beta_a E_a} \, , \tag{3.44}$$

where $E_a = p_a^2/2m_a$ is the kinetic energy and $\beta_a = 1/T_a$ is the inverse temperature in energy units. The thermal wave length for species a is defined by

$$\lambda_a = \hbar \left(\frac{2\pi\beta_a}{m_a} \right)^{1/2} . \tag{3.45}$$

Consequently, the spatial density of the kinetic energy of species a is given by

$$\mathcal{E}_a = \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{p_a^2}{2m_a} f_a(\mathbf{p}_a, t) \, , \tag{3.46}$$

where f_a is the corresponding distribution function.

Suppose now that the rate of change in the distribution function f_a is determined by some kinetic equation

$$\frac{\partial f_a}{\partial t} + \mathbf{v}_a \cdot \boldsymbol{\nabla}_x f_a = \sum_b K_{ab}[f] \, , \tag{3.47}$$

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where $\boldsymbol{\nabla}_x$ is the ν-dimensional gradient in position space, $\mathbf{v}_a = \mathbf{p}_a/m_a$ is the particle velocity, and K_{ab} is a scattering kernel between particles of type a and type b. When the distribution is spatially uniform we may set the convective term to zero, $\mathbf{v}_a \cdot \boldsymbol{\nabla}_x f_a = 0$, in which case the time rate of change in the kinetic-energy density (3.46) of the a-species is given by

$$\frac{d\mathcal{E}_a}{dt} = \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{p_a^2}{2m_a} \frac{\partial f_a}{\partial t}(\mathbf{p}_a, t) = \sum_b \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{p_a^2}{2m_a} K_{ab}[f] \ . \tag{3.48}$$

We can therefore identify the rate of change in the kinetic-energy density of species a, resulting from its Coulomb interactions with species b, by the expression

$$\frac{d\mathcal{E}_{ab}}{dt} = \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{p_a^2}{2m_a} K_{ab}[f] \ . \tag{3.49}$$

Since we are taking each species to be in thermal equilibrium with itself, but not necessarily with the other species, each species b is characterized by a unique temperature T_b. The rate $d\mathcal{E}_{ab}/dt$ is therefore proportional to the temperature difference between these species, and we write

$$\frac{d\mathcal{E}_{ab}}{dt} = -\mathcal{C}_{ab}(T_a - T_b) \ , \tag{3.50}$$

where $\mathcal{C}_{ab}$ is called the rate coefficient. By performing the integrals in (3.49) exactly, and then comparing with (3.50), we may calculate the coefficients $\mathcal{C}_{ab}$ exactly. If the ions are at a common temperature T_I, then it is more convenient to calculate the rate coefficient between the electrons and the collective set of ions, the coefficient $\mathcal{C}_{e\mathrm{I}} = \sum_i C_{ei}$ of (1.1).

2. The Boltzmann Equation

The derivation of the Boltzmann equation presented in Section 3.3 of Ref. [10] goes through unscathed in a general number of dimensions, and the scattering kernel is completely finite when $\nu > 3$. The derivation breaks down in $\nu \leq 3$ (that is to say, for $\nu = 1, 2, 3$), because in these dimensions the scattering kernel for the Coulomb interaction diverges. This behavior for the Boltzmann equation arises because the Coulomb interaction emphasizes the short distance physics when $\nu > 3$, while the scattering kernel of the Boltzmann equation is designed to capture such short distance physics. I will write the Boltzmann equation in schematic form as

$$\frac{\partial f_a}{\partial t} + \mathbf{v}_a \cdot \boldsymbol{\nabla}_x f_a = \sum_b B_{ab}[f] \quad : \nu > 3 \ , \tag{3.51}$$

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or in explicit form by writing the full scattering kernel as

$$
B_{ab}[f] = \int \frac{d^\nu p_b}{(2\pi\hbar)^\nu} \, d\Omega \, |\mathbf{v}_b - \mathbf{v}_a| \frac{d\sigma_{ab}}{d\Omega} \left\{ f_a(\mathbf{p}'_a) f_b(\mathbf{p}'_b) - f_a(\mathbf{p}_a) f_b(\mathbf{p}_b) \right\}
$$

$$
(2\pi\hbar)^\nu \, \delta^\nu\!\left(\mathbf{p}'_a + \mathbf{p}'_b - \mathbf{p}_a - \mathbf{p}_b \right) (2\pi\hbar)\delta\!\left(E'_a + E'_b - E_a - E_b \right) , \tag{3.52}
$$

with $E'_c = p_c'^2/2m_c$ and $E_c = p_c^2/2m_c$. See Fig. 5 for an explanation of the cross section $d\sigma_{ab}$ in ν spatial dimensions. We can include the quantum effects of two-body scattering, to the order in g to which we are working, by replacing the classical cross section by the corresponding quantum cross section defined in (3.23). It then becomes necessary to calculate the quantum transition amplitude $T(ab \to a'b') \equiv T_{a'b';ab}$, and rewriting (3.23) in the form

$$
|\mathbf{v}_b - \mathbf{v}_a| \, d\sigma_{ab} = \left| T_{a'b';ab} \right|^2 \frac{d^\nu p'_a}{(2\pi\hbar)^\nu} \frac{d^\nu p'_b}{(2\pi\hbar)^\nu} , \tag{3.53}
$$

one can then include quantum effects by using the scattering kernel

$$
B_{ab}[f] = \int \frac{d^\nu p'_a}{(2\pi\hbar)^\nu} \frac{d^\nu p'_b}{(2\pi\hbar)^\nu} \frac{d^\nu p_b}{(2\pi\hbar)^\nu} \left| T_{a'b';ab} \right|^2 \left\{ f_a(\mathbf{p}'_a) f_b(\mathbf{p}'_b) - f_a(\mathbf{p}_a) f_b(\mathbf{p}_b) \right\}
$$

$$
(2\pi\hbar)^\nu \, \delta^\nu\!\left(\mathbf{p}'_a + \mathbf{p}'_b - \mathbf{p}_a - \mathbf{p}_b \right) (2\pi\hbar)\delta\!\left(E'_a + E'_b - E_a - E_b \right) . \tag{3.54}
$$

For simplicity, in Sec. IV A we shall use the Born approximation (3.32) for the transition amplitude, which corresponds to taking the extreme quantum limit. When $\nu > 3$, expression (3.49) allows us to write the rate of change of the energy density resulting from the now finite Boltzmann equation as

$$
\frac{d\mathcal{E}^>_{ab}}{dt} = \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{p_a^2}{2m_a} \, B_{ab}[f] \qquad : \quad \nu > 3 . \tag{3.55}
$$

I have used a "greater than" superscript to remind us that we should calculate (3.55) in dimensions greater than three.

3. The Lenard-Balescu Equation

In dimensions less than three one finds a complementary situation to the Boltzmann equation, namely, the derivation of the Lenard-Balescu equation is rigorous and completely finite when $\nu < 3$. This is because the long distance physics of the Coulomb potential is dominant in dimensions $\nu < 3$, and the Lenard-Balescu equation is designed to capture such long distance physics. I will write the Lenard-Balescu equation in schematic form as

$$
\frac{\partial f_a}{\partial t} + \mathbf{v}_a \cdot \boldsymbol{\nabla}_x f_a = \sum_b L_{ab}[f] \qquad : \nu < 3 , \tag{3.56}
$$

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where the kernel of the ν-dimensional Lenard-Balescu equation is the obvious generalization from three dimensions,

$$L_{ab}[f] = \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{d^\nu p_b}{(2\pi\hbar)^\nu} \frac{d^\nu k}{(2\pi)^\nu} \, \boldsymbol{\nabla}_{p_a} \cdot \mathbf{k} \left| \frac{e_a \, e_b}{k^2 \, \epsilon(\mathbf{k}, \mathbf{v}_a \cdot \mathbf{k})} \right|^2 \pi \, \delta(\mathbf{v}_a \cdot \mathbf{k} - \mathbf{v}_b \cdot \mathbf{k})$$

$$\left\{ \mathbf{k} \cdot \boldsymbol{\nabla}_{p_b} - \mathbf{k} \cdot \boldsymbol{\nabla}_{p_a} \right\} f_a(\mathbf{p}_a) \, f_b(\mathbf{p}_b) \,, \quad (3.57)$$

where $\boldsymbol{\nabla}_{p_c}$ is the ν-dimensional momentum gradient. Reference [11] shows that the dielectric function of a weakly to moderately coupled plasma is given by

$$\epsilon(\mathbf{k}, \omega) = 1 + \sum_c \frac{e_c^2}{k^2} \int \frac{d^\nu \mathbf{p}_c}{(2\pi\hbar)^\nu} \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_c + i\eta} \, \mathbf{k} \cdot \boldsymbol{\nabla}_{p_c} f_c(\mathbf{p}_c) \,, \quad (3.58)$$

where the prescription $\eta \to 0^+$ is implicit and defines the correct retarded response. We can use (3.58) in (3.57) to the order in g to which we are working. The sum in (3.58) is over all plasma components, and the velocity $\mathbf{v}_c = \mathbf{p}_c/m_c$ appearing in the denominator is really an an integration variable. Therefore, when $\nu < 3$, the rate (3.49) allows us to express

$$\frac{d\mathcal{E}_{ab}^<}{dt} = \int \frac{d^\nu p}{(2\pi\hbar)^\nu} \frac{p^2}{2m_a} \, L_{ab}[f] \quad : \quad \nu < 3 \,. \quad (3.59)$$

I have used a "less than" superscript to remind us that we should calculate (3.59) in dimensions less than three.

It is convenient to express the dielectric function in terms of a complex function $F(v)$ defined by the relation

$$k^2 \, \epsilon(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}) = k^2 + F(v \cos\theta) \,, \quad (3.60)$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{v}$. The engineering unit of the argument of F is velocity, while the unit of F itself is wave-number squared. Expressions (3.58) and (3.60) imply the dispersion relation

$$F(u) = \int_{-\infty}^{+\infty} dv \, \frac{\rho_{\text{total}}(v)}{v - u - i\eta} \,, \quad (3.61)$$

where the limit $\eta \to 0^+$ is understood, with the spectral weight being defined by

$$\rho_{\text{total}}(v) = \sum_c \rho_c(v) \quad (3.62)$$

$$\rho_c(v) = \kappa_c^2 \left(\frac{\beta_c m_c}{2\pi} \right)^{1/2} v \, \exp\left\{ -\frac{1}{2}\beta_c m_c v^2 \right\} \,. \quad (3.63)$$

For future use, we shall require the convenient relations

$$F(-v) = F^*(v) \quad (3.64)$$

$$\text{Im} \, F(v) = \frac{1}{2i}\left[F(v) - F^*(v) \right] = \pi \, \rho_{\text{total}}(v) \,. \quad (3.65)$$

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While we shall not require the real part of F, nor is there space to compute this function, for completeness I shall record it here:

$$\operatorname{Re} F(v) = \sum_b \kappa_b^2 \left[1 - 2\sqrt{\frac{\beta_b m_b}{2}}\, v \, \operatorname{daw}\left(\sqrt{\frac{\beta_b m_b}{2}}\, v \right) \right] , \qquad (3.66)$$

where the Dawson integral is defined by

$$\operatorname{daw}(x) \equiv \int_0^x dy \, e^{y^2 - x^2} . \qquad (3.67)$$

E. Calculating the Rate

Returning to three dimensions for a moment, and dropping the species index on (3.46) for simplicity, the rate of change in the kinetic energy density is simply given by

$$\frac{d\mathcal{E}}{dt} = \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{p^2}{2m} \frac{\partial f}{\partial t}(\mathbf{p}, t) \qquad (3.68)$$

The problem with a straightforward evaluation of (3.68) in three dimensions is that any potentially tractable kinetic equation gives a logarithmically divergent result for the rate, either at short or long distances, depending on the deficiencies of the particular kinetic equation in hand. In principle, however, calculating the rate is a well defined procedure: it is the approximation scheme employed in finding the requisite distribution function that introduces a divergence. In other words, if one *knew* the exact single-particle distribution function $f(\mathbf{p}, t)$, then the rate (3.68) would be finite. However, the one-point function $f_1 = f(\mathbf{p}, t)$ of the BBGKY hierarchy can be known exactly only by solving the entire set of coupled multi-particle correlation functions exactly, an impossible feat. Hence, we must approximate the exact distribution function by one obtained through truncating the BBGKY kinetic equations. Even worse, the truncation process is rather subjective in that it depends upon the type of physics one deems important; for example, truncation to the Boltzmann equation is only useful if we can neglect long distance correlations (which, in this problem, we cannot[9]). Conversely, truncation to the Lenard-Balescu equation captures the long distance physics, but misses the short distance physics. In a nutshell, then, our problem is the following: to calculate the rate, we require the exact distribution function of the full hierarchy of kinetic equations, a problem we cannot hope to solve without a Quantum Computer or

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[9] In reducing BBGKY to the Boltzmann equation, we make the approximation that two-body collisions are uncorrelated, thereby allowing the replacement $f_2 \to f_1 \cdot f_1$ in the scattering kernel. However, the correlations described by f_2 and higher act back on f_1 to render the integrals in (3.68) finite at long distances (and such correlations are neglected by the Boltzmann equation).

a Mentat; we must therefore truncate the kinetic equations, but this does violence to either the short or long distance physics, thereby introducing spurious divergences.

A hint out of this Catch 22 comes from the following observation. The truncation problem only occurs for the *Coulomb potential*, and only then in *three* spatial dimensions. As we saw in (3.28), the Coulomb potential in ν dimensions is $V_\nu(r) = C_\nu\, e_a e_b / r^{\nu-2}$, where $C_\nu = \Gamma(\nu/2-1)/4\pi^{\nu/2}$ is a geometric constant, and this form of the potential renders the scattering kernels of the Boltzmann and Lenard-Balescu equations finite, except for the single case $\nu = 3$ (ironically, the case of interest). Let us therefore start with the exact BBGKY hierarchy in ν spatial dimensions, with the understanding that taking $\nu \neq 3$ is a regulating procedure, and that we must eventually return to three dimensions. This procedure, however, is robust enough to capture the correct physics as the limit $\nu \to 3$ is taken.

1. Reduction of BBGKY

In the exact same manner as in the last few section, one can generalize the BBGKY hierarchy to an arbitrary number of dimensions. Furthermore, as we discussed in Lecture I [3], when the number of spatial dimension is greater than three, BBGKY reduces to the Boltzmann equation (3.51) and (3.52) to leading order in the plasma coupling g. Conversely, when $\nu < 3$ the BBGKY hierarchy reduces to the Lenard-Balescu equation (3.56) and (3.57) to leading order in g. As discussed in the previous section, these reduced kinetic equations (Boltzmann and Lenard-Balescu) are finite in their respective dimensional regimes. In other words, besides rendering the truncation process finite, the physical utility of keeping the dimension of space arbitrary is that in dimensions greater than three, the leading order in g behavior of BBGKY is just the finite ν-dimensional Boltzmann equation:

$$\text{BBGKY in } \nu > 3 \ \Rightarrow \ \frac{\partial f_a}{\partial t} + \mathbf{v}_a \cdot \boldsymbol{\nabla}_x f_a \ = \ \sum_b B_{ab}[f] \quad \text{to LO in } g \ , \qquad (3.69)$$

where $B_{ab}[f]$ is given by (3.52). As discussed at length in Lecture I, expression (3.69) is the point at which the *physics* of dimensional continuation enters the calculation: dimensions greater than three select for short distance physics. For $\nu > 3$, the rate of energy transport from plasma species a to species b is therefore given by the finite expression (3.55). Turning now to dimensions less than three, we have seen in Lecture I that the leading order in g behavior of BBGKY reduces to the Lenard-Balescu equation,

$$\text{BBGKY in } \nu < 3 \ \Rightarrow \ \frac{\partial f_a}{\partial t} + \mathbf{v}_a \cdot \boldsymbol{\nabla}_x f_a \ = \ \sum_b L_{ab}[f] \quad \text{to LO in } g \ , \qquad (3.70)$$

where the finite ν-dimensional scattering kernel $L_{ab}[f]$ is given by (3.57). Again, the physical content of dimensional continuation enters at this stage: dimensions less than three select for long distance physics. This dimensional reduction of BBGKY is illustrated in Fig. 7.
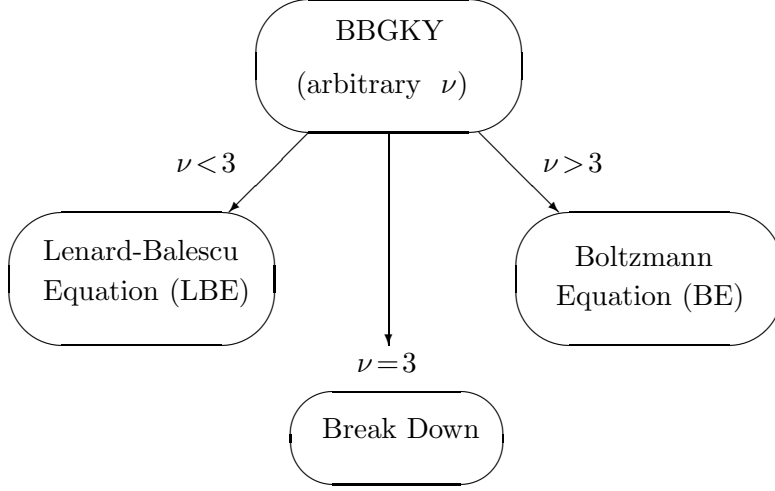
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FIG. 7: For $\nu > 3$ the "textbook derivation" of the Boltzmann equation for a Coulomb potential is rigorous; furthermore, the BBGKY hierarchy reduces to the Boltzmann equation to leading order in g. A similar reduction from the BBGKY hierarchy holds for the Lenard-Balescu equation in $\nu < 3$, and the "textbook derivation" is also rigorous in these dimensions. In $\nu = 3$, the derivations of the Boltzmann and Lenard-Balescu equations break down for the Coulomb potential.

From the rate equations (3.47)–(3.49), in dimensions $\nu > 3$ expression (3.69) gives the rate of energy transport from species a to species b as

$$\frac{d\mathcal{E}_{ab}^>}{dt} = \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{p_a^2}{2m_a} B_{ab}[f] \text{ to LO in } g , \qquad (3.71)$$

which, as we shall see, takes the form

$$\frac{d\mathcal{E}_{ab}^>}{dt} = H(\nu) \frac{g^2}{\nu - 3} + \mathcal{O}(\nu - 3) \quad \text{to LO in } g \text{ when } \nu > 3 . \qquad (3.72)$$

We have omitted the species indices from $H(\nu)$ for simplicity, and we shall calculate this quantity in Section IV A. Rather than a logarithmically divergent result, we obtain a finite answer involving a simple pole of the form $1/(\nu - 3)$, which of course is the origin of the divergence in three dimensions. Since we will eventually return to $\nu = 3$, there is no need to calculate the $\mathcal{O}(\nu - 3)$ terms in (3.72), as these terms vanish when $\nu \to 3$. Similarly, from (3.70) the corresponding rate in energy transport from species a to b is therefore

$$\frac{d\mathcal{E}_{ab}^<}{dt} = \int \frac{d^\nu p_a}{(2\pi\hbar)^\nu} \frac{p_a^2}{2m_a} L_{ab}[f] \quad \text{to LO in } g . \qquad (3.73)$$

From the calculation in Sec. IV B, we shall find

$$\frac{d\mathcal{E}_{ab}^<}{dt} = G(\nu) \frac{g^{\nu-1}}{3 - \nu} + \mathcal{O}(3 - \nu) \quad \text{to LO in } g \text{ when } \nu < 3 , \qquad (3.74)$$

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where we have omitted the species indices from $G(\nu)$ to save writing. Note that the leading behavior of both (3.72) and (3.74) is formally of order g^2 in three dimensions, illustrating that neither short nor long distance physics dominates in $\nu = 3$, but rather, that ultraviolet and infrared length scales contribute to the same order in three dimensions. It is a property of the Coulomb potential itself that $\nu = 3$ is the fulcrum around which the short and long distance physics pivot. Using the appropriate Coulomb potential (3.28) for $V_\nu(r)$ in the scattering kernels of (3.71) and (3.73), the integrals now converge, and they are calculated exactly in Sections 7 and 8 of BPS [2]. In these notes we shall calculate them in Sections IV A and IV B.

2. Obtaining Next-to-Leading Order from Leading Order

In three dimensions, or in the limit $\nu \to 3$, we still are still plagued by the long and short distance divergences from the simple poles in (3.72) and (3.74), a problem we must now confront if we are to obtain a meaningful result. As described in Lecture I, to compare the rates (3.72) and (3.74), we must analytically continue one or the other to a *common* value of the dimension ν. Analytically continuing the spatial dimension makes sense because we can view the quantities $d\mathcal{E}_{ab}^{>}/dt$ and $d\mathcal{E}_{ab}^{<}/dt$ as functions of a complex parameter ν, even though they were only calculated for positive integer values of ν. This is analogous to analytically continuing the factorial function on the positive integers to the Gamma function on the complex plane. For definiteness, I will analytically continue (3.74) to $\nu > 3$, in which case the g-dependence becomes subleading relative to the g^2 dependence of (3.72). The analytic continuation of (3.74) takes the same functional form for any $\nu \in \mathbb{C}$, but in this section I will write the analytic continuation as

$$\frac{d\mathcal{E}_{ab}^{<}}{dt} = -G(\nu) \frac{g^{2+(\nu-3)}}{\nu - 3} + \mathcal{O}(\nu - 3) \quad \text{to NLO in } g \text{ when } \nu > 3 . \tag{3.75}$$

Since we are now working in the regime $\nu - 3 > 0$, I have written the exponent of g in a form to emphasize that $g^2 \gg g^{\nu-1}$ when $g \ll 1$ and $\nu > 3$. There are no terms with powers of g intermediate g^2 and $g^{\nu-1}$, so the analytic continuation of $d\mathcal{E}_{ab}^{<}/dt$ to parameters $\nu > 3$ is not only subleading in g, but it is indeed next-to-leading order relative to (3.72). This is illustrated in Fig. 8.

In the rates (3.72) and (3.74), we need to work consistently only to linear order in the small parameter $\epsilon = \nu - 3$; therefore, we should expand $H(\nu)$ and $G(\nu)$ to first order in ϵ, allowing us to write

$$H(\nu) = -A + \epsilon H_1 + \mathcal{O}(\epsilon^2) \tag{3.76}$$

$$G(\nu) = -A + \epsilon G_1 + \mathcal{O}(\epsilon^2) . \tag{3.77}$$

analytically continue
around the $\nu=3$ pole

$$d\mathcal{E}^<_{ab}/dt \qquad\qquad d\mathcal{E}^<_{ab}/dt$$

$$g^{\nu-1} = g^{2-(3-\nu)} \qquad 3 \qquad g^{\nu-1} = g^{2+(\nu-3)}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$g^{2-|\nu-3|} \qquad\qquad g^{2+|\nu-3|}$$

LO: large when $g \ll 1$ \qquad NLO: small when $g \ll 1$
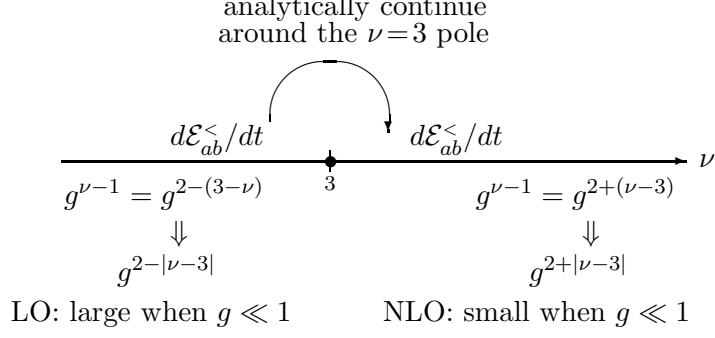
FIG. 8: The analytic continuation of $d\mathcal{E}^<_{ab}/dt$ from $\nu < 3$ to the region $\nu > 3$ in the complex ν-plane. The same expression can be used for $d\mathcal{E}^<_{ab}/dt$ throughout the complex plane since the pole at $\nu = 3$ can easily be avoided. The quantity $d\mathcal{E}^<_{ab}/dt \sim g^{2+(\nu-3)}$ is leading order in g for $\nu < 3$. However, upon analytically continuing to $\nu > 3$ we find that $d\mathcal{E}^<_{ab}/dt \sim g^{2+|\nu-3|}$ is next-to-leading order in g relative to $d\mathcal{E}^>_{ab}/dt \sim g^2$.

It is crucially important here that $H(\nu)$ and $G(\nu)$ give the same value at $\nu = 3$, a term that I have called A in (3.76) and (3.77), otherwise the divergent poles will not cancel. In Section IV, we shall calculate the coefficients $H(\nu)$ and $G(\nu)$, and we will indeed explicitly see that $H(\nu = 3)$ and $G(\nu = 3)$ are equal. We will also calculate $H_1 \equiv H'(\nu = 3)$ and $G_1 \equiv G'(\nu=3)$ in closed form, thereby providing an exact result for the rate to leading and next-to-leading order in g.

Finally, upon writing $g^\epsilon = \exp\{\epsilon \ln g\}$ in (3.75), and expanding to first order in ϵ, we find

$$\frac{g^\epsilon}{\epsilon} = \frac{1}{\epsilon} + \ln g + \mathcal{O}(\epsilon) . \tag{3.78}$$

This is where the nonanalyticity in g arises, *i.e.* the $\ln g$ term, and we can now express the rates (3.72) and (3.75) as

$$\frac{d\mathcal{E}^>_{ab}}{dt} = -\frac{A}{\nu - 3} g^2 + H_1 g^2 + \mathcal{O}(\nu - 3; g^3) \qquad\qquad \nu > 3 \tag{3.79}$$

$$\frac{d\mathcal{E}^<_{ab}}{dt} = \frac{A}{\nu - 3} g^2 - G_1 g^2 - A g^2 \ln g + \mathcal{O}(\nu - 3; g^3) \qquad \nu > 3 . \tag{3.80}$$

These expressions hold in the common dimension $\nu > 3$, and to obtain the leading and next-to-leading order result in three dimensions, we add and take the limit:

$$\frac{d\mathcal{E}_{ab}}{dt} = \lim_{\nu \to 3^+} \left[\frac{d\mathcal{E}^>_{ab}}{dt} + \frac{d\mathcal{E}^<_{ab}}{dt} \right] + \mathcal{O}(g^3) = -Ag^2 \ln g + Bg^2 + \mathcal{O}(g^3) , \tag{3.81}$$

with $B = H_1 - G_1$. Compare this with the rate (2.2), or the alternative expression (2.3). In this way, BPS calculated the energy exchange rate from Coulomb interactions between plasma species, accurate to leading order and next-to-leading order in g.

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IV. CALCULATING THE RATE TO SUBLEADING ORDER

We now perform the real calculation of the energy exchange rate between electrons and ions. BPS [2] considered the general case, finding $\mathcal{C}_{ab}$ in (3.50) for any collection of plasma species in any quantum regime. In these notes, however, I will only consider the extreme quantum limit valid at high temperatures. This is the case of most general interest, and it is also the case in which the algebra simplifies considerably. We shall also take the electrons to be in equilibrium with themselves at temperature T_e and the ions in equilibrium with themselves at temperature T_{I}, another situation of general interest. Upon summing over the ions, the rate equation then becomes

$$\frac{d\mathcal{E}_{e\mathrm{I}}}{dt} = -\mathcal{C}_{e\mathrm{I}} \left(T_e - T_{\mathrm{I}}\right) , \tag{4.1}$$

where the collective rate coefficient that we shall calculate is

$$\mathcal{C}_{e\mathrm{I}} = \sum_i \mathcal{C}_{ei} . \tag{4.2}$$

A. Boltzmann-Equation: Short-Distance Physics

We now work in $\nu > 3$ dimensions where the short-distance physics dominates. To calculate the rate of change of the electron distribution, we will employ the Boltzmann equation with two-body quantum effects in the Born approximation. The rate of energy exchange from the electrons to the ions is therefore,

$$\frac{\partial \mathcal{E}_{e\mathrm{I}}^{>}}{\partial t} = \int \frac{d^{\nu} p_e}{(2\pi\hbar)^{\nu}} \frac{p_e^2}{2m_e} \frac{\partial f_e(\mathbf{p}_e)}{\partial t} = \sum_i \int \frac{d^{\nu} p_e}{(2\pi\hbar)^{\nu}} \frac{p_e^2}{2m_e} B_{ei}[f] , \tag{4.3}$$

where we have taken $a = e$ and $b = i$ in (3.51) and (3.54), and the electron and ion distribution functions are given by (3.44). Using the crossing symmetries $\mathbf{p}_e \leftrightarrow \mathbf{p}_e'$ and $\mathbf{p}_i \leftrightarrow \mathbf{p}_i'$ of the scattering amplitude T_{B} in (3.54), the rate of energy exchange from the electrons to an ion species i can be written

$$\begin{aligned}
\frac{\partial \mathcal{E}_{ei}^{>}}{\partial t} &= \int \frac{d^{\nu} p_e'}{(2\pi\hbar)^{\nu}} \frac{d^{\nu} p_i'}{(2\pi\hbar)^{\nu}} \frac{d^{\nu} p_e}{(2\pi\hbar)^{\nu}} \frac{d^{\nu} p_i}{(2\pi\hbar)^{\nu}} \left|T_{\mathrm{B}}\right|^2 \frac{p_e'^2 - p_e^2}{2m_e} f_e(\mathbf{p}_e) f_i(\mathbf{p}_i) \\
&\quad (2\pi\hbar)^{\nu} \delta^{\nu}\!\left(\mathbf{p}_i' + \mathbf{p}_e' - \mathbf{p}_i - \mathbf{p}_e\right) (2\pi\hbar) \delta\!\left(\frac{p_e'^2 - p_e^2}{2m_e} + \frac{p_i'^2 - p_i^2}{2m_i}\right) ,
\end{aligned} \tag{4.4}$$

where we dropped the gradient term in (3.54) because of spatial uniformity. Summing over all ions in (4.4) gives the total rate (4.3). We define the momentum transfer $\mathbf{q}$ and the average momentum $\bar{\mathbf{p}}$ of the initial and final electron momentum,

$$\mathbf{q} \equiv \mathbf{p}_e' - \mathbf{p}_e = \mathbf{p}_i - \mathbf{p}_i' \tag{4.5}$$

$$\bar{\mathbf{p}} \equiv \frac{1}{2}\left[\mathbf{p}_e' + \mathbf{p}_e\right] . \tag{4.6}$$

Upon performing the p_i'-integral using the momentum conserving delta-function to set

$$\mathbf{p}_i' = \mathbf{p}_i + \mathbf{p}_e - \mathbf{p}_e' = \mathbf{p}_i - \mathbf{q} \; , \tag{4.7}$$

and expressing the electron momenta in terms of $\mathbf{q}$ and $\bar{\mathbf{p}}$,

$$\mathbf{p}_e' \; = \; \bar{\mathbf{p}} + \frac{1}{2}\mathbf{q} \qquad \mathbf{p}_e = \bar{\mathbf{p}} - \frac{1}{2}\mathbf{q} \; , \tag{4.8}$$

we can simplify (4.4) to read

$$\frac{\partial \mathcal{E}_{ei}^{>}}{\partial t} = \int \frac{d^\nu \bar{p}}{(2\pi\hbar)^\nu} \frac{d^\nu q}{(2\pi\hbar)^\nu} \left| T_{\mathrm{B}}(q) \right|^2 \frac{m_i}{m_e} \bar{\mathbf{p}}{\cdot}\mathbf{q} \; f_e(\bar{\mathbf{p}} - \mathbf{q}/2) \times$$

$$\int \frac{d^\nu p_i}{(2\pi\hbar)^\nu} (2\pi\hbar)\delta\left(\mathbf{p}_i{\cdot}\mathbf{q} - \frac{m_i}{2m_e} \bar{\mathbf{p}}{\cdot}\mathbf{q} - \frac{1}{2}q^2 \right) f_i(\mathbf{p}_i) \; . \tag{4.9}$$

We have used the fact that the energy conserving delta function and the energy loss factor become,

$$\frac{p_e'^2 - p_e^2}{2m_e} \; = \; \frac{1}{m_e} \bar{\mathbf{p}}{\cdot}\mathbf{q} \tag{4.10}$$

$$\delta\left(\frac{p_e'^2 - p_e^2}{2m_e} + \frac{p_i'^2 - p_i^2}{2m_i} \right) \; = \; \delta\left(\frac{\bar{\mathbf{p}}{\cdot}\mathbf{q}}{m_e} - \frac{\mathbf{p}_i{\cdot}\mathbf{q}}{2m_i} + \frac{q^2}{2m_i} \right) \; . \tag{4.11}$$

We now perform the p_i-integration. Since we will find a similar integral in the next section, I will perform a more general calculation here. There will be times when we need to integrate a Gaussian and a delta-function, which I will write as

$$\int \frac{d^\nu p_b}{(2\pi\hbar)^\nu} \, \delta(\hat{\mathbf{k}}{\cdot}\mathbf{v}_b - V) \, e^{-\beta_b E_b} \; = \; \hbar\beta_b \left(\frac{1}{\lambda_b} \right)^{\nu+1} e^{-\frac{1}{2}\beta_b m_b V^2} = \frac{m_b}{2\pi\hbar} \lambda^{1-\nu} \, e^{-\frac{1}{2}\beta_b m_b V^2} , \tag{4.12}$$

where $\hat{\mathbf{k}}$ is a fixed unit vector (typically another integration variable), and V is a scalar independent of $p_b = m_b\mathbf{v}_b$. The integral (4.12) will be required in several places throughout the text, so we will perform the calculation here. Since $\mathbb{V} = \hat{\mathbf{v}}_b \cdot \hat{\mathbf{k}}$ defines the component of $\mathbf{v}_b$ parallel to $\hat{\mathbf{k}}$, will decompose the integration variables $\mathbf{v}_b$ into parallel and normal components

$$\mathbf{v}_b = \mathbf{v}_\perp + (\mathbf{v}_b{\cdot}\hat{\mathbf{k}}) \, \hat{\mathbf{k}} = \mathbf{v}_\perp + \mathbb{V} \, \hat{\mathbf{k}} \; . \tag{4.13}$$

Since $v_b^2 = v_\perp^2 + \mathbb{V}^2$, we can write

$$\int \frac{d^\nu p_b}{(2\pi\hbar)^\nu} \, \delta(\mathbf{v}_b{\cdot}\hat{\mathbf{k}} - V) \, e^{-\ell\beta_b \, p_\ell^2 / 2m_b}$$

$$= \left(\frac{m_b}{2\pi\hbar} \right)^\nu \int d^{\nu-1} v_\perp e^{-\ell\frac{1}{2}\beta_b m_b v_\perp^2} \cdot \int_{-\infty}^{\infty} d\mathbb{V} \, \delta(\mathbb{V} - V) \, e^{-\ell\frac{1}{2}\beta_b m_b \mathbb{V}^2} \tag{4.14}$$

$$= \left(\frac{m_b}{2\pi\hbar} \right)^\nu \left(\frac{2\pi}{\ell\beta_b m_b} \right)^{(\nu-1)/2} \cdot e^{-\ell\frac{1}{2}\beta_b m_b V^2} \; , \tag{4.15}$$

and this yields expression (4.12). As an application, we will frequently run across an integral

$$\int \frac{d^\nu p_b}{(2\pi\hbar)^\nu} \, \delta(\mathbf{v}_b \cdot \hat{\mathbf{k}} - V) \, f_b(p_b) = n_b \left(\frac{\beta_b m_b}{2\pi} \right)^{1/2} e^{-\frac{1}{2}\beta_b m_b V^2} \, , \tag{4.16}$$

which follows directly from (4.12). Using the integral (4.16), and taking (3.44) for $f_e(\bar{\mathbf{p}} - \mathbf{q}/2)$ gives

$$\frac{\partial \mathcal{E}_{ei}^>}{\partial t} = \frac{n_e n_i}{m_e} \, (2\pi\beta_i m_i)^{1/2} \, \lambda_e^\nu \int \frac{d^\nu \bar{p}}{(2\pi\hbar)^\nu} \frac{d^\nu q}{(2\pi\hbar)^\nu} \, \hbar \left| T_{\mathrm{B}}(q) \right|^2 \bar{\mathbf{p}} \cdot \hat{\mathbf{q}}$$

$$\exp\left\{ -\frac{\beta_i m_i}{2m_e^2} \left[(\bar{\mathbf{p}} \cdot \hat{\mathbf{q}})^2 + \frac{m_e}{m_i} \left(1 - \frac{\beta_e}{\beta_i} \right) q \, \bar{\mathbf{p}} \cdot \hat{\mathbf{q}} \right] - \frac{\beta_e}{2m_e} \left[\bar{p}^2 + \left(1 + \frac{m_e \beta_i}{m_i \beta_e} \right) \frac{q^2}{4} \right] \right\}. \tag{4.17}$$

When $\beta_e = \beta_i$, the linear term in the exponential involving $\bar{\mathbf{p}} \cdot \mathbf{q}$ vanishes. The integrand is even in both $\bar{\mathbf{p}}$ and $\bar{\mathbf{q}}$, except for the prefactor $\bar{\mathbf{p}} \cdot \hat{\mathbf{q}}$; therefore, keeping $\mathbf{q}$ fixed and integrating over $\bar{\mathbf{p}}$ gives zero,

$$\frac{\partial \mathcal{E}_{ei}^>}{\partial t} \bigg|_{\beta_e = \beta_i} = 0 \, ,$$

as it should for equal electron and ion temperatures.

We now examine the general case when the electron and ion temperatures differ. Completing the square for the terms in the first square-brackets suggests changing variables to

$$\bar{\mathbf{p}}' = \bar{\mathbf{p}} + \frac{m_e}{m_i} \left(1 - \frac{\beta_e}{\beta_i} \right) \frac{\mathbf{q}}{2} \, , \tag{4.18}$$

and dropping q^2-terms that are down by relative factors of m_e/m_i gives

$$\frac{\partial \mathcal{E}_{ei}^>}{\partial t} = \frac{n_e n_i}{m_e} \, (2\pi\beta_i m_i)^{1/2} \, \lambda_e^\nu \int \frac{d^\nu \bar{p}'}{(2\pi\hbar)^\nu} \frac{d^\nu q}{(2\pi\hbar)^\nu} \, \hbar \left| T_{\mathrm{B}}(q) \right|^2 \left[\bar{\mathbf{p}}' \cdot \hat{\mathbf{q}} + \frac{m_e}{m_i} \left(\frac{\beta_e}{\beta_i} - 1 \right) \frac{q}{2} \right]$$

$$\exp\left\{ -\frac{\beta_i m_i}{2m_e^2} \, (\bar{\mathbf{p}}' \cdot \hat{\mathbf{q}})^2 - \frac{\beta_e}{2m_e} \left[\bar{p}_\perp'^2 + (\bar{\mathbf{p}}' \cdot \hat{\mathbf{q}})^2 + \frac{q^2}{4} \right] \right\} \, , \tag{4.19}$$

where we have expanded the new integration variable $\bar{\mathbf{p}}'$ into normal and parallel components

$$\bar{\mathbf{p}}' = \bar{\mathbf{p}}_\perp' + \bar{\mathbb{P}}' \hat{\mathbf{q}} \quad \text{with} \quad \bar{\mathbb{P}}' = \bar{\mathbf{p}}' \cdot \hat{\mathbf{q}} \, . \tag{4.20}$$

Finally, we note that the term $(\bar{\mathbf{p}}' \cdot \hat{\mathbf{q}})^2$ in the square brackets of the exponential is down by a factor m_e/m_i relative to the first such term, and we can write (4.19) as

$$\frac{\partial \mathcal{E}_{ei}^>}{\partial t} = \frac{n_e n_i}{m_e} \, (2\pi\beta_i m_i)^{1/2} \, \lambda_e^\nu \int \frac{d\mathbb{P}}{2\pi\hbar} \int \frac{d^{\nu-1}\bar{p}_\perp'}{(2\pi\hbar)^{\nu-1}} \frac{d^\nu q}{(2\pi\hbar)^\nu} \, \hbar \left| T_{\mathrm{B}}(q) \right|^2$$

$$\left[\mathbb{P}' + \frac{m_e}{m_i} \left(\frac{\beta_e}{\beta_i} - 1 \right) \frac{q}{2} \right] \exp\left\{ -\frac{\beta_i m_i}{2m_e^2} \, \mathbb{P}'^2 - \frac{\beta_e}{2m_e} \left[\bar{p}_\perp'^2 + \frac{q^2}{4} \right] \right\} \, . \tag{4.21}$$

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The linear $\mathbb{P}'$ term in the prefactor integrates to zero, and we arrive at

$$\frac{\partial \mathcal{E}_{ei}^>}{\partial t} = \frac{n_e\, n_i}{2m_i} \, (2\pi\beta_i m_i)^{1/2} \, \lambda_e^\nu \left(\frac{\beta_e}{\beta_i} - 1\right) \int \frac{d\mathbb{P}}{2\pi\hbar} \int \frac{d^{\nu-1}\bar{p}'_\perp}{(2\pi\hbar)^{\nu-1}} \frac{d^\nu q}{(2\pi\hbar)^\nu} \, \hbar \left|T_{\mathrm{B}}(q)\right|^2$$

$$q \, \exp\left\{ -\frac{\beta_i m_i}{2m_e^2} \mathbb{P}'^2 - \frac{\beta_e}{2m_e}\left[\bar{p}'^2_\perp + \frac{q^2}{4}\right]\right\} . \quad (4.22)$$

The integrand is now Gaussian in the variables $\mathbb{P}'$ and $\bar{\mathbf{p}}'^2_\perp$, and we find

$$(2\pi\beta_i m_i)^{1/2} \int_{-\infty}^\infty \frac{d\bar{\mathbb{P}}'}{2\pi\hbar} \exp\left\{ -\frac{\beta_i m_i}{2m_e^2} \bar{\mathbb{P}}'^2 \right\} = \frac{m_e}{\hbar} \quad (4.23)$$

$$\lambda_e^\nu \int \frac{d^{\nu-1}\bar{p}'_\perp}{(2\pi\hbar)^{\nu-1}} \exp\left\{ \frac{\beta_e}{2m_e} \bar{p}'^2_\perp \right\} = \lambda_e . \quad (4.24)$$

Substituting back into (4.22) gives

$$\frac{\partial \mathcal{E}_{ei}^>}{\partial t} = \frac{n_e\, n_i}{2m_i} \frac{\lambda_e\, m_e}{\hbar} \underbrace{\left(\frac{\beta_e}{\beta_i} - 1\right)}_{\beta_e\left(T_i - T_e\right)} \int \frac{d^\nu q}{(2\pi\hbar)^\nu} \, \hbar \left|T_{\mathrm{B}}(q)\right|^2 q \, \exp\left\{ -\frac{\beta_e}{2m_e} \frac{q^2}{4} \right\} . \quad (4.25)$$

Up to this point, the two-body scattering amplitude T could have been general, but we now explicitly employ the Born approximation

$$T_{\mathrm{B}} = \hbar \frac{ee_i}{q^2} , \quad (4.26)$$

so that

$$\frac{\partial \mathcal{E}_{ei}^>}{\partial t} = \kappa_e^2 \left(\frac{1}{2}\lambda_e\, m_e\, \hbar^2\right)\omega_i^2\left(T_i - T_e\right) \int \frac{d^\nu q}{(2\pi\hbar)^\nu} \frac{1}{q^3} \exp\left\{ -\frac{\beta_e}{2m_e} \frac{q^2}{4} \right\} , \quad (4.27)$$

where the ion plasma frequency is $\omega_i^2 = e_i^2\, n_i/m_i$ and the electron Debye wave-number is $\kappa_e^2 = e^2 n_e/T_e$.

Another expression we will encounter is the gamma-function,

$$\Gamma(z) = \int_0^\infty du\, u^{z-1}\, e^{-u} \quad \mathrm{Re}(z) > 0 , \quad (4.28)$$

The gamma-function (4.28) provides a nice trick for calculating integrals of the form

$$a^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty ds\, s^{n-1}\, e^{-a\, s} . \quad (4.29)$$

In doing calculations, we will often need to take the norm of a vector to some power, such as $|\mathbf{q}|^{-(\nu-3)/2}$. For example, we can use (4.29) to exponentiate the norm into a more easily handled Gaussian by writing $|\mathbf{q}| = (\mathbf{q} \cdot \mathbf{q})^{1/2}$, so that

$$|\mathbf{q}|^{-m} = \frac{1}{\Gamma(m/2)} \int_0^\infty ds\, s^{m/2-1}\, e^{-s\, \mathbf{q} \cdot \mathbf{q}} . \quad (4.30)$$

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We can now perform the q-integral to give

$$\int \frac{d^\nu q}{(2\pi\hbar)^\nu} \, q^{-3} \, e^{-\beta_e q^2/8m_e} = \frac{\Omega_{\nu-1}}{(2\pi\hbar)^\nu} \int_0^\infty dq \, q^{\nu-4} e^{-\beta_e q^2/8m_e} \qquad : s = \frac{\beta_e}{8m_e} \, q^2 \qquad (4.31)$$

$$= \frac{1}{(2\pi\hbar)^\nu} \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)} \left(\frac{8m_e}{\beta_e}\right)^{(\nu-3)/2} \frac{1}{2} \int_0^\infty ds \, s^{(\nu-5)/2} \, e^{-s} \quad (4.32)$$

$$= \frac{1}{4\pi^2 \, \hbar^3} \frac{\Gamma(3/2)}{\Gamma(\nu/2)} \left(\frac{4}{\lambda_e^2}\right)^{(\nu-3)/2} \Gamma\left(\frac{\nu-3}{2}\right) , \qquad (4.33)$$

where $\Gamma(3/2) = \sqrt{\pi}/2$, and substituting this back into (4.27) allows us to express

$$\frac{\partial \mathcal{E}_{ei}^>}{\partial t} = \frac{\kappa_e^2}{2\pi} \left(\frac{\lambda_e \, m_e}{2\pi\hbar}\right) \omega_i^2 \left(T_i - T_e\right) \frac{\Gamma(3/2)}{\Gamma(\nu/2)} \frac{1}{2} \left(\frac{4}{\lambda_e^2}\right)^{(\nu-3)/2} \Gamma\left(\frac{\nu-3}{2}\right) . \qquad (4.34)$$

Note that

$$\frac{\lambda_e \, m_e}{2\pi\hbar} = \left(\frac{\beta_e m_e}{2\pi}\right)^{1/2} , \qquad (4.35)$$

and therefore the rate coefficient becomes, upon dividing by the temperature difference and then summing over the ion components,

$$\mathcal{C}_{eI}^> = \frac{\kappa_e^2}{2\pi} \, \omega_I^2 \left(\frac{\beta_e m_e}{2\pi}\right)^{1/2} \frac{\Gamma(3/2)}{\Gamma(\nu/2)} \frac{1}{2} \left(\frac{4}{\lambda_e^2}\right)^{(\nu-3)/2} \Gamma\left(\frac{\nu-3}{2}\right) \qquad (4.36)$$

B. Lenard-Balescu Equation: Long-Distance Physics

We now wish to calculate the leading order long-distance physics by working in spatial dimensions $\nu < 3$. The rate of energy exchange from the electrons to the ions is

$$\frac{\partial \mathcal{E}_e^<}{\partial t} = 2 \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu} \frac{p_e^2}{2m_e} \frac{\partial f_e(\mathbf{p}_e)}{\partial t} , \qquad (4.37)$$

and from (3.57) we find the rate of energy exchange between a spatially uniform distribution of electrons and ion species i,

$$\frac{\partial \mathcal{E}_{ei}^<}{\partial t} = -2 \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu} \frac{d^\nu p_i}{(2\pi\hbar)^\nu} \frac{d^\nu k}{(2\pi)^\nu} \frac{p_e^2}{2m_e} \, \boldsymbol{\nabla}_{p_e} \cdot \mathbf{k} \left| \frac{e \, e_i}{k^2 \, \epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_e)} \right|^2 \pi\delta(\hat{\mathbf{k}}\cdot\mathbf{v}_e - \hat{\mathbf{k}}\cdot\mathbf{v}_i)$$

$$\left(\hat{\mathbf{k}}\cdot\mathbf{v}_i \, \beta_i - \hat{\mathbf{k}}\cdot\mathbf{v}_e \, \beta_e\right) f_e(\mathbf{p}_e) f_i(\mathbf{p}_i) , \quad (4.38)$$

where we have used the distribution (3.44) to write $\boldsymbol{\nabla}_{p_b} f_b(\mathbf{p}_b) = -\beta_b \, \mathbf{v}_b \, f_b(\mathbf{p}_b)$. Integrating $\mathbf{p}_e$ by parts using $\mathbf{k} \cdot \boldsymbol{\nabla}_{p_e}(p_e^2/2m_e) = \mathbf{k} \cdot \mathbf{v}_e$, and integrating over the ion distribution with (4.16) gives

$$\frac{\partial \mathcal{E}_{ei}^<}{\partial t} = 2\left(\frac{\beta_e}{\beta_i} - 1\right) \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu} \frac{d^\nu k}{(2\pi)^\nu} \frac{\pi \, e^2 \, \mathbf{k}\cdot\mathbf{v}_e \, \rho_i(\hat{\mathbf{k}} \cdot \mathbf{v}_e)}{\left|k^2 + F(\hat{\mathbf{k}}\cdot\mathbf{v}_e)\right|^2} \, f_e(\mathbf{p}_e) , \qquad (4.39)$$

where we have used (3.60) and (3.63). When the electron and ion temperatures are equal, $\beta_e = \beta_i$, we see that the rate vanishes, as it must. Inserting unity in the form

$$\int_{-\infty}^{\infty} dv\, \delta(v - \hat{\mathbf{k}} \cdot \mathbf{v}_e) = 1 \tag{4.40}$$

allows us to express the rate as

$$\frac{\partial \mathcal{E}_{ei}^{<}}{\partial t} = 2\left(\frac{\beta_e}{\beta_i} - 1\right) \int_{-\infty}^{\infty} dv \int \frac{d^\nu p_e}{(2\pi\hbar)^\nu} \frac{d^\nu k}{(2\pi)^\nu} \frac{\pi e^2 k v \rho_i(v)}{\left|k^2 + F(v)\right|^2} \delta(v - \hat{\mathbf{k}} \cdot \mathbf{v}_e) f_e(\mathbf{p}_e) \,, \tag{4.41}$$

and upon performing the electron momentum integrals with (4.16), we find

$$\frac{\partial \mathcal{E}_{ei}^{<}}{\partial t} = 2\, \kappa_e^2 \left(\frac{\beta_e m_e}{2\pi}\right)^{1/2} \left(T_{\mathrm{I}} - T_e\right) \pi \int \frac{d^\nu k}{(2\pi)^\nu} k \int_{-\infty}^{\infty} dv\, \frac{v \rho_i(v)}{\left|k^2 + F(v)\right|^2} e^{-\frac{1}{2}\beta_e m_e v^2} \,, \tag{4.42}$$

where we have used (1.3) for κ_e.

For individual ion species i, the v-integral can only be performed numerically; however, if we sum over all ion species we can perform the integral by completing a contour in the complex v-plane [2]:

$$\lim_{m_e \to 0} \sum_i \int_{-\infty}^{\infty} dv\, \frac{v \rho_i(v)}{\left|k^2 + F(v)\right|^2} e^{-\frac{1}{2}\ell\beta_e m_e v^2} = \frac{\omega_{\mathrm{I}}^2}{(k^2 + \kappa_e^2)^2} \quad \text{with} \quad \omega_{\mathrm{I}}^2 = \sum_i \omega_i^2 \,. \tag{4.43}$$

Summing over the ions in (4.42) therefore allows us to express

$$\frac{\partial \mathcal{E}_{eI}^{<}}{\partial t} = 2\, \kappa_e^2\, \omega_{\mathrm{I}}^2 \left(\frac{\beta_e m_e}{2\pi}\right)^{1/2} \left(T_{\mathrm{I}} - T_e\right) \pi \int \frac{d^\nu k}{(2\pi)^\nu} \frac{k}{(k^2 + \kappa_e^2)^2} \,. \tag{4.44}$$

Note that the k-integral

$$\mathbb{J} = \int \frac{d^\nu k}{(2\pi)^\nu} \frac{k}{(k^2 + \kappa_e^2)^2} \tag{4.45}$$

converges both in the UV and IR. This is because the κ_e^2 term in the denominator provides a long-distance cutoff (small values of k), while the condition $\nu < 3$ provides short-distance convergence (large values of k). This integral can be performed by first converting to hyperspherical coordinates and then changing variables by $k^2 = \kappa_e^2\, t$:

$$\mathbb{J} = \frac{\Omega_{\nu-1}}{(2\pi)^\nu} \int_0^{\infty} k^{\nu-1} dk\, \frac{k}{(k^2 + \kappa_e^2)^2} = \frac{\kappa_e^{\nu-3}}{2} \frac{\Omega_{\nu-1}}{(2\pi)^\nu} \int_0^{\infty} dt\, t^{\nu-1/2}(t+1)^{-2} \,, \tag{4.46}$$

where the solid angle integrals $\Omega_{\nu-1}$ are given in (3.26). The t-integral can be recognized as the Beta function, defined in (3.41) with $x = (\nu+1)/2$ and $y = (3-\nu)/2$. Inserting the appropriate Gamma functions and factors of π gives

$$\int \frac{d^\nu k}{(2\pi)^\nu} \frac{k}{(k^2 + \kappa_e^2)^2} = \frac{1}{4\pi^2} \frac{\Gamma(3/2)}{\Gamma(\nu/2)} \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(\frac{3-\nu}{2}\right) \left(\frac{\kappa_e^2}{4\pi}\right)^{(\nu-3)/2} \,, \tag{4.47}$$

where $\Gamma(3/2) = \sqrt{\pi}/2$. Substituting (4.47) back (4.44) gives

$$\frac{\partial \mathcal{E}_{\mathrm{eI}}^<}{\partial t} = 2\,\kappa_e^2\,\omega_{\mathrm{I}}^2 \left(\frac{\beta_e m_e}{2\pi}\right)^{1/2} \left(T_{\mathrm{I}} - T_e\right) \frac{1}{4\pi} \frac{\Gamma(3/2)}{\Gamma(\nu/2)} \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(\frac{3-\nu}{2}\right) \left(\frac{\kappa_e^2}{4\pi}\right)^{(\nu-3)/2} , \quad (4.48)$$

or the rate coefficient

$$\mathcal{C}_{\mathrm{eI}}^< = \frac{\kappa_e^2}{2\pi}\,\omega_{\mathrm{I}}^2 \left(\frac{\beta_e m_e}{2\pi}\right)^{1/2} \frac{\Gamma(3/2)}{\Gamma(\nu/2)} \frac{1}{2} \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(\frac{3-\nu}{2}\right) \left(\frac{\kappa_e^2}{4\pi}\right)^{(\nu-3)/2} . \quad (4.49)$$

C.   Combining the Leading and Next-to-Leading Order Terms

Recall that the rate coefficient in $\nu < 3$ and $\nu > 3$ takes the exact analytic form

$$\mathcal{C}_{\mathrm{eI}}^> = \frac{\kappa_e^2}{2\pi}\,\omega_{\mathrm{I}}^2 \left(\frac{\beta_e m_e}{2\pi}\right)^{1/2} \frac{\Gamma(3/2)}{\Gamma(\nu/2)} \frac{1}{2} \Gamma\left(\frac{\nu-3}{2}\right) \left(\frac{4}{\lambda_e^2}\right)^{(\nu-3)/2} \quad (4.50)$$

$$\mathcal{C}_{\mathrm{eI}}^< = \frac{\kappa_e^2}{2\pi}\,\omega_{\mathrm{I}}^2 \left(\frac{\beta_e m_e}{2\pi}\right)^{1/2} \frac{\Gamma(3/2)}{\Gamma(\nu/2)} \frac{1}{2} \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(\frac{3-\nu}{2}\right) \left(\frac{\kappa_e^2}{4\pi}\right)^{(\nu-3)/2} . \quad (4.51)$$

The gamma functions in the above expressions can be expanded in the parameter $\epsilon = \nu - 3$:

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \quad (4.52)$$

$$\Gamma(1+\epsilon) = 1 - \gamma\,\epsilon + \mathcal{O}(\epsilon^2) . \quad (4.53)$$

We often need to multiply a term of the form $A^{\nu-3}$ by a pole $1/(\nu-3)$. This will produce a pole term *and* a finite contribution. In fact, this is the origin of the coefficient under the logarithm, and since this point is so important, I will reiterate it once again. For ease of notation let $\epsilon = \nu - 3$, so that the $\nu \to 3$ limit is the same as the $\epsilon \to 0$ limit. In any calculation we can therefore drop terms $\mathcal{O}(\epsilon)$ and higher; however, we must be careful and drop such terms too soon. This is because an order ϵ term could multiply a pole term of the form $1/\epsilon$, thereby giving a finite nonzero result in the limit $\epsilon \to 0$. The following example illustrates this point. Let us consider the product of A^ϵ with the pole $1/\epsilon$. We first expand A^ϵ in powers of ϵ as follows

$$A^\epsilon = \exp\{\ln A^\epsilon\} = \exp\{\epsilon \ln A\} = 1 + \epsilon \ln A + \mathcal{O}(\epsilon^2) . \quad (4.54)$$

Upon multiplying this expression by the pole we find

$$\frac{A^\epsilon}{\epsilon} = \frac{1}{\epsilon} + \ln A + \mathcal{O}(\epsilon) . \quad (4.55)$$

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Therefore, upon using expression (4.55) in (4.50) and (4.51), we find

$$\Gamma\left(\frac{\nu-3}{2}\right)\left[\frac{4}{\lambda_e^2}\right]^{(\nu-3)/2} = \frac{2}{\nu-3}+\ln\left\{\frac{4}{\lambda_e^2}\right\}-\gamma \qquad (4.56)$$

$$\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{3-\nu}{2}\right)\left[\frac{\kappa_e^2}{4\pi}\right]^{(\nu-3)/2} = -\frac{2}{\nu-3}-\ln\left\{\frac{\kappa_e^2}{4\pi}\right\}-1 \ . \qquad (4.57)$$

This gives the rate coefficient

$$\mathcal{C}_{eI}=\mathcal{C}_{eI}^{>}+\mathcal{C}_{eI}^{<}=\left[\frac{\kappa_e^2}{2\pi}\,\omega_I^2\,\left(\frac{\beta_e m_e}{2\pi}\right)^{1/2}\right]\cdot\frac{1}{2}\left[\ln\left\{\frac{16\pi}{\kappa_e^2\,\lambda_e^2}\right\}-\gamma-1\right] \ . \qquad (4.58)$$

Note that the pole terms have canceled, rendering a finite result accurate to leading and next-to-leading order in g. As we have seen, the next-to-leading order term gives the exact coefficient under the logarithm (including the term $-\gamma-1$). The argument of the logarithm can be expressed as

$$\frac{16\pi}{\lambda_e^2\,\kappa_e^2}=\frac{8\,m_e T_e}{\hbar^2}\,\frac{T_e}{e^2\,n_e}=\frac{8\,T_e^2}{\hbar^2\,\omega_e^2} \ , \qquad (4.59)$$

and therefore the rate coefficient takes the form

$$\mathcal{C}_{eI} \;=\; \frac{\kappa_e^2}{2\pi}\,\omega_I^2\,\sqrt{\frac{\beta_e m_e}{2\pi}}\,\ln\Lambda_{\rm BPS} \qquad (4.60)$$

$$\ln\Lambda_{\rm BPS} \;=\; \frac{1}{2}\left[\ln\left\{\frac{8T_e^2}{\hbar^2\omega_e^2}\right\}-\gamma-1\right] \ . \qquad (4.61)$$

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